M55: Exercise sheet 5

1. Let X be a set. What is the coarsest topology on X for which all singleton sets are closed?
2. Show that a metric space is normal.
3. Let A, B be compact subsets of a topological space X. Show that $A \cup B$ is compact.
4. (a) Let $(F_n)_{n\in\mathbb{N}}$ be a family of non-empty closed subsets of a compact topological space X such that, $F_{n+1} \subset F_n$, for all $n \in \mathbb{N}$.
   Prove that
   $$\bigcap_{n\in\mathbb{N}} F_n \neq \emptyset.$$
   (b) Let $f : X \to X$ be a continuous map of a compact Hausdorff topological space into itself. Show there is a non-empty closed subset A of X such that $f(A) = A$.
5. Let X be a set equipped with the co-finite topology. Show that any subset of X is compact.
   Deduce that $\mathbb{R}$ with the Zariski topology has compact subsets which are not closed.
6. Show that a compact Hausdorff space is normal (and so T_4).
7. Recall the Moore plane Γ.
   (a) Show that any subset of the x-axis is closed in Γ.
   (b) Show that Γ is regular.
   (c) Show that Γ is not normal[1].
8. Show that there is no Hausdorff topology on $[0, 1]$ which is strictly coarser than the metric topology.
   Hint: Contemplate the identity map...

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[1]This is sufficiently difficult that I offer a small prize for the first correct solution!

Home page: `http://www.maths.bath.ac.uk/~feb/math0055.html`

M55: Exercise sheet 5—Solutions

1. The co-finite topology is coarsest topology for which all singleton sets are closed: indeed, any topology for which singleton sets are closed has all finite sets closed (they are finite unions of singleton sets!) and so contains the co-finite topology.

2. Let $C, D \subset X$ be disjoint closed subsets of a metric space. For $x \in C$, $x \in X \setminus D$ which is open so there is $\varepsilon_x > 0$ with $B_{\varepsilon_x}(x) \subset X \setminus C$. Similarly, for $y \in D$, there is $\varepsilon_y > 0$ such that $B_{\varepsilon_y}(y) \subset X \setminus C$.
   Now define open sets by
   $$U = \bigcup_{x \in C} B_{\varepsilon_x/2}(x) \qquad V = \bigcup_{y \in D} B_{\varepsilon_y/2}(y).$$
   Then $C \subset U \subset X \setminus D$ and $D \subset V \subset X \setminus C$ and all we have to do is see that U and V are disjoint. However, if $z \in U \cap V$, there is $x \in C$ and $y \in D$ such that $d(x, z) < \varepsilon_x/2$ and $d(z, y) < \varepsilon_y/2$. The triangle inequality now gives that $d(x, y) < \max\{\varepsilon_x, \varepsilon_y\}$. Without loss of generality, suppose $\varepsilon_x \geq \varepsilon_y$. Then we have $y \in B_{\varepsilon_x}(x) \cap D$: a palpable contradiction.

3. Let $\{U_\alpha\}_{\alpha \in I}$ be an open cover for $A \cup B$. Thus each U_α is open in X and $\bigcup_{\alpha \in I} U_\alpha \supset A \cup B$. Thus $\{U_\alpha\}_{\alpha \in I}$ is also an open cover of A and so there is a finite set $J_A \subset I$ with $\bigcup_{\alpha \in J_A} U_\alpha \supset A$. Similarly, there is another finite set $J_B \subset I$ with $\bigcup_{\alpha \in J_B} U_\alpha \supset B$. Now set $J = J_A \cup J_B$. Then $\bigcup_{\alpha \in J} U_\alpha \supset A \cup B$ whence $\{U_\alpha\}_{\alpha \in J}$ is a finite subcover for $A \cup B$ and so $A \cup B$ is compact.

4. (a) Suppose, for a contradiction, that $\bigcap_{n \in \mathbb{N}} F_n = \emptyset$. Taking complements and using De Morgan's Laws gives us $X = \bigcup_{n \in \mathbb{N}} (X \setminus F_n)$ so that $\{X \setminus F_n\}_{n \in \mathbb{N}}$ is an open cover of X. Thus there is a finite subcover $X \setminus F_{n_1}, \dots, X \setminus F_{n_k}$ with $n_1 < \dots < n_k$. Thus $\bigcup_{i=1}^k (X \setminus F_{n_i} = X$ or taking complements $\bigcap_{i=1}^k F_{n_i} = \emptyset$. But, since each $F_{n+1} \subset F_n$, $\bigcap_{i=1}^k F_{n_i} = F_{n_k} \neq \emptyset$. This is a contradiction!
   (b) Let f^n denote the n-times composition of f with itself:
   $$f^n = \underbrace{f \circ \dots \circ f}_{n \text{ times}}$$
   and set $F_n = f^n(X)$. Now X is compact and f^n continuous so that F_n is compact and therefore closed since X is Hausdorff. Further, $F_{n+1} = f^n(f(X)) \subset f^n(X) = F_n$ so we are in the situation of part (a). So set $A = \bigcap_{n \in \mathbb{N}} F_n$ which is non-empty by part (a). I claim that $f(A) = A$.
   For this, first note that if $f(F_n) = F_{n+1}$ so that
   $$f(A) \subset \bigcap_{n \in \mathbb{N}} f(F_n) = \bigcap_{n \in \mathbb{N}} F_{n+1} = A.$$
   For equality, let $a \in A$ and contemplate $f^{-1}(\{a\}) \cap F_n$: these are closed since $\{a\}$ is closed and f is continuous (X is Hausdorff), nested since the F_n are and non-empty: for each n, $a \in F_{n+1}$ and so there is an $x_n \in F_n$ such that $f(x_n) = a$. Thus we are once again in the situation of part (a) and so can conclude that
   $$\emptyset \neq \bigcap_{n \in \mathbb{N}} \left(f^{-1}(\{a\}) \cap F_n\right) = f^{-1}(\{a\}) \cap A.$$
   Thus there is some $x \in a$ such that $f(x) = a$ and so $f(A) = A$ as required.

5. Let $A \subset X$ and $\{U_\alpha\}_{\alpha \in I}$ be an open cover for A. Let U_{α_0} be a non-empty element of the cover. Then $X \setminus U_{\alpha_0}$ and so $A \setminus U_{\alpha_0}$ is finite. Write $A \setminus U_{\alpha_0} = \{a_1, \dots, a_n\}$ and for each i between 1 and n, choose U_{α_i} containing a_i. Then $U_{\alpha_0}, \dots, U_{\alpha_n}$ covers A and so is a finite subcover. Thus A is compact.
   When $\mathbb{R}$ is equipped with this topology, there are many non-closed sets (for example, $\mathbb{Z}$) but all of these are compact by what we have just proved.

6. Let C, D be disjoint closed (and therefore compact) subsets of X and fix $x \in C$. Then, for each $y \in D$, there are disjoint open sets U_y, V_y with $x \in U_y$ and $y \in V_y$. Then $\{V_y\}_{y\in D}$ is an open cover of D and so there is a finite subcover $V_{y_1}, \dots, V_{y_n}$. Now set $U_x = \bigcap_{1\leq i\leq n} U_{y_n}$ and $W_x = \bigcup_{1\leq i\leq n} V_{y_n}$. Then $x \in U_x$, $D \subset W_x$ and U_x, W_x are disjoint open sets.
In particular, $\{U_x\}_{x\in C}$ is an open cover of C so that there is a finite subcover $U_{x_1}, \dots, U_{x_m}$. Set $U = \bigcup_{1\leq i\leq m} U_{x_i}$ and $V = \bigcap_{1\leq i\leq m} W_{x_i}$ to get disjoint open sets with $U \supset C$ and $V \supset D$ whence X is normal and so (being Hausdorff) T_4.

7. (a) First notice that the x-axis is closed in Γ: its complement is the union of all the ε-balls in $\{(x, y) : y > 0\}$. Second, recall that the induced topology on the x-axis is the discrete topology so that all subsets of the x-axis are closed with respect to the subspace topology on the x-axis. Thus any subset of the x-axis is the intersection of the x-axis with a closed subset of Γ and so is itself closed in Γ.
(b) Let $C \subset \Gamma$ be closed and $p \in \Gamma \setminus C$. Now $\Gamma \setminus C$ is open so there is a basic open set B with $p \in B \subset \Gamma \setminus C$. If p is in the strict upper half plane then we may take B to be an ε-ball about p and then set $U = B_{\varepsilon/2}(p)$, $V = \{q \in \Gamma : \|q - p\| > \varepsilon/2\}$ to get disjoint open sets with $p \in U$ and $C \subset V$.
If, on the other hand, p lies on the x-axis, then B is of the form $A \cup \{p\}$ with A an ε-ball tangent to the x-axis at p. Let A' be the ball of radius $\varepsilon/2$ tangent to the x-axis at p and let D be its (metric) closure. Then $U = A' \cup \{p\}$ and $V = \Gamma \setminus D$ are disjoint open sets with $p \in U$ and $C \subset V$.
(c) This is quite tough: one must find disjoint closed sets C, D in Γ such that whenever U, V are open sets with $C \subset U$ and $D \subset V$ then $U \cap V \neq \emptyset$. We take $C = \mathbb{Q}$ and $D = \mathbb{R} \setminus \mathbb{Q}$: these are disjoint subsets of the x-axis and so closed by part (a). Suppose that U, V are disjoint open sets with $U \supset C$ and $V \supset D$. Then, for $p \in \mathbb{R}$, there is a ball tangent to the x-axis at p of radius $r(p) > 0$ which lies entirely in U or V according to whether p is rational or not. Now for the clever bit: define subsets $A_n \subset \mathbb{R}$, $n \in \mathbb{N}$, by

$$
\begin{aligned}
A_{2n} &= \{q \in \mathbb{Q} : r(q) > 1/n\} \\
A_{2n-1} &= \{p \notin \mathbb{Q} : r(p) > 1/n\}.
\end{aligned}
$$

The complete metric space $\mathbb{R}$ is the union of the A_n and so the Baire Category Theorem tells us that at least one A_n is not nowhere dense, that is, there is some n for which the closure of A_n contains an interval (a, b). Without loss of generality, assume that n is even (and so consists of rationals) and choose some irrational $p \in (a, b)$. Then there is a sequence (q_k) in A_n converging to p. I claim that the $r(p)$-ball tangent to p must intersect some $r(q_k)$-ball tangent to q_k which contradicts the disjointness of U, V. For this, choose $0 < y < \min\{1/n, r(p)\}$ so that (p, y) is in the $r(p)$-ball tangent to p. Then, as $k \to \infty$,

$$\|(p, y) - (q_k, 1/n)\| \to |y - 1/n| < 1/n.$$

Thus, for k large enough, (p, y) lies in the $1/n$-ball tangent to the x-axis at q_k and so in the $r(q_k)$-ball tangent to the x-axis at q_k. This completes the proof.

8. Denote the metric topology on $[0, 1]$ by $\mathscr{T}_d$ and suppose that $\mathscr{T}$ is a coarser Hausdorff topology on $[0, 1]$. Then $\mathrm{id} : (X, \mathscr{T}_d) \to (X, \mathscr{T})$ is a continuous bijection from a compact space to a Hausdorff one and so is a homeomorphism. Thus $\mathscr{T}_d = \mathscr{T}$.