M55: Exercise sheet 4

1. Let $A \subset X$ and $B \subset Y$ be closed subsets of topological spaces X and Y. Show that $A \times B$ is closed (with respect to the product topology, of course).
2. Suppose X and Y are topological spaces and that y is a point of Y. Show that X is homeomorphic to the subspace $X \times \{y\}$ of $X \times Y$.
3. A map $f : X \to Y$ of topological spaces is *open* if $f(U)$ is open in Y for all U open in X.
   Show that the projection $\pi_1 : X \times Y \to X$ is open.
   Is the image of a closed set necessarily closed?
4. Show that the product topology is the unique topology on $X \times Y$ with the property that for any map $h = (f, g) : Z \to X \times Y$ of a topological space Z, h is continuous if and only if $f : Z \to X$ and $g : Z \to Y$ are continuous.
5. Show that if X is Hausdorff then every singleton set $\{x\}$, $x \in X$, is closed.
6. Show that X is Hausdorff if and only if, whenever $x \neq y$, there is an open set U with $x \in U$ and $y \notin \overline{U}$.
7. Is the Moore plane Hausdorff?

March 14, 2003

M55: Exercise sheet 4—Solutions

1. We have no choice but to show that $(X \times Y) \setminus (A \times B)$ is open, that is, a union of basic open sets.
   Now $(x, y) \notin A \times B$ if and only if either $x \notin A$ or $y \notin B$. That is,
   $$(X \times Y) \setminus (A \times B) = (X \setminus A) \times Y \cup X \times (Y \setminus B).$$
   Since $X \setminus A$ is open in X and $Y \setminus B$ is open in Y, this last is a union of basic (in fact, subbasic) open sets and so open.

2. Define $f : X \to X \times \{y\}$ by $f(x) = (x, y)$. Now f is bijective with continuous inverse $\pi_1|_{X \times \{y\}}$. For the continuity of f, note that $\pi_1 \circ f = \mathrm{id}_X$ and $\pi_2 \circ f$ is constant so that both components of f are continuous whence f is continuous as a map into $X \times Y$ and thus as a map into $X \times \{y\}$ with the subspace topology.

3. Any open set U in $X \times Y$ is a union of basic open sets: $U = \bigcup_{\alpha \in I} U_\alpha \times V_\alpha$ and then
   $$\pi_1(U) = \pi_1(\bigcup_{\alpha \in I} U_\alpha \times V_\alpha) = \bigcup_{\alpha \in I} \pi_1(U_\alpha \times V_\alpha) = \bigcup_{\alpha \in I} U_\alpha$$
   which is a union of open sets in X and so open.
   The converse is false: let $C = \{(x, y) \in \mathbb{R}^2 : x > 0 \text{ and } y = 1/x\}$. This is a closed subset of $\mathbb{R}^2$ (use sequences!) but $\pi_1(C) = (0, \infty)$ which is not open in $\mathbb{R}$.

4. We have already proved in lectures that the product topology has this property. For uniqueness, denote the product topology by $\mathscr{T}$ and suppose that $\mathscr{S}$ is another topology with the same property.
   First take $Z = (X \times Y, \mathscr{S})$ and $h = \mathrm{id}_{X \times Y} = (\pi_1, \pi_2)$. By the property, since $h = \mathrm{id}_{X \times Y}$ is continuous so are the π_i but this means, by the fact that $\mathscr{T}$ also has the property that $\mathrm{id}_{X \times Y} : (X \times Y, \mathscr{S}) \to (X \times Y, \mathscr{T})$ is continuous. Thus $\mathscr{T} \subset \mathscr{S}$.
   Conversely, take $Z = (X \times Y, \mathscr{T})$ and once more $h = \mathrm{id}_{X \times Y} = (\pi_1, \pi_2)$. We know that the π_i are continuous with respect to $\mathscr{T}$ so the property gives that $\mathrm{id}_{X \times Y} : (X \times Y, \mathscr{T}) \to (X \times Y, \mathscr{S})$ is continuous. Thus $\mathscr{S} \subset \mathscr{T}$ so that $\mathscr{S} = \mathscr{T}$ and we are done.

5. For $y \neq x$, there are disjoint open sets U_y, V_y with $x \in U_y$ and $y \in V_y$. In particular, $x \notin V_y$ or, otherwise said, $V_y \subset X \setminus \{x\}$. Thus
   $$X \setminus \{x\} = \bigcup_{y \neq x} V_y$$
   which is open so that $\{x\}$ is closed.

6. Suppose X is Hausdorff and $x \neq y$. Then there are disjoint open U, V with $x \in U$ and $y \in V$. Since $U \cap V = \emptyset$, $U \subset X \setminus V$ which last is closed so that $\overline{U} \subset X \setminus V$. In particular, $y \notin \overline{U}$.
   For the converse, if there is an open U with $x \in U$ and $y \notin \overline{U}$ then put $V = X \setminus \overline{U}$. V is open (since $\overline{U}$ is closed), disjoint from U (since $U \subset \overline{U}$) and certainly contains y. Thus X is Hausdorff.

7. Yes. Draw pictures to see that basic open sets build out of sufficiently small ε-balls will separate any two points.