M55: Exercise sheet 3

1. Let A be a closed subset of a topological space X and $F \subset A$. Show that F is closed in A if and only if F is closed in X.

2. Let $B \subset A \subset X$. Let C be the closure of B in X and C_A the closure of B in A. Show that $C_A = A \cap C$.

3. (Another patching type theorem) Let $A, B \subset X$ be closed subsets of a topological space X such that $X = A \cup B$. Equip both A and B with the subspace topology. Let $f : X \to Y$ be a map into another topological space Y.
   Show that f is continuous if and only if both $f_{|A} : A \to Y$ and $f_{|B} : B \to Y$ are continuous (with respect to the subspace topologies).
   Use this to prove that $f : [-1, 1] \to \mathbb{R}$ defined by
   $$f(x) = \begin{cases} -x & \text{if } x \in [-1, 0]; \\ x & \text{if } x \in [0, 1]; \end{cases}$$
   is continuous.

4. Let X be a topological space with base $\mathscr{B}$ and $A \subset X$. Let $\mathscr{B}_A = \{A \cap B : B \in \mathscr{B}\}$.
   (a) Show that $\mathscr{B}_A$ is a base for a topology on A.
   (b) What topology on A is it?

5. (**The Moore plane**) Let Γ denote the closed upper half plane $\{(x, y) \in \mathbb{R}^2 : y \geq 0\}$. Let $\mathscr{B}$ denote all ε-balls lying in the region $\{(x, y) : y > 0\}$ along with all sets of the form $\{z\} \cup A$ with z a point on the x-axis and A an open ball in the upper half plane tangent to the x-axis at z.
   (a) Show that $\mathscr{B}$ is the base for a topology on Γ.
   (b) What is the induced topology on the set $\{(x, y) : y > 0\}$?
   (c) What is the induced topology on the half-line $\{(0, y) : y \geq 0\}$?
   (d) What is the induced topology on the x-axis?

March 7, 2003

M55: Exercise sheet 3—Solutions

1. If F is closed in A then $F = A \cap G$ with G closed in X. Thus F is the intersection of two closed subsets of X and so is itself a closed subset of X.
   Conversely if F is closed in X and $F \subset A$, then $F = A \cap F$ is closed in A (note that closedness of A is not needed for this bit).

2. First note that $A \cap C$ is closed in A and contains B whence $C_A \subset A \cap C$. For the converse, we use a characterisation of C_A given in lectures. Let $x \in A \cap C$ and $x \in U$ with U open in A. We need to show that $U \cap B \neq \emptyset$. Now $U = A \cap V$ with V open in X and since $x \in C$, we know that $B \cap V \neq \emptyset$. Now, since $B \subset A$, $B \cap V = B \cap (V \cap A) = B \cap U$ so that $B \cap U \neq \emptyset$ and $x \in C_A$.

3. If f is continuous then certainly $f_{|A}$ and $f_{|B}$ are continuous being the composition of f and the (continuous) inclusions of A, B into X.
   For the converse, it is a good idea to use the closed set formulation of continuity: so let $C \subset Y$ be closed and try to show that $f^{-1}(C)$ is closed in X.
   Now $f^{-1}(C) = (f^{-1}(C) \cap A) \cup (f^{-1}(C) \cap B)$ since $X = A \cup B$. Also $f^{-1}(C) \cap A = f_{|A}^{-1}(C)$ which is closed in A (since $f_{|A}$ is continuous) and so closed in X by question 1. Similarly, $f^{-1}(C) \cap B$ is closed in X whence $f^{-1}(C)$ is closed in X being a union of two sets both closed in X. Thus f is continuous.
   For $f : [-1, 1] \to \mathbb{R}$ defined as above, $f_{|[-1,0]}$ is clearly continuous—it is the restriction of the notoriously continuous $x \mapsto -x$ and similarly $f_{|[0,1]}$ is continuous whence, by what we have just proved, f is continuous since $[-1, 0]$ and $[0, 1]$ are closed in $\mathbb{R}$ and so in $[-1, 1]$.

4. (a) We check the axioms for a base: first, since $\mathscr{B}$ is a base, any $x \in A$ lies in some $B \in \mathscr{B}$ and so in $A \cap B \in \mathscr{B}_A$. Second, let $A \cap B_1, A \cap B_2 \in \mathscr{B}_A$ with the $B_i \in \mathscr{B}$. If $x \in (A \cap B_1) \cap (A \cap B_2)$ then $x \in B_1 \cap B_2$ and there is $B_3 \in \mathscr{B}$ with $x \in B_3 \subset B_1 \cap B_2$. Thus $x \in A \cap B_3 \subset (A \cap B_1) \cap (A \cap B_2)$ and $A \cap B_3 \in \mathscr{B}_A$ so that $\mathscr{B}_A$ is a base for a topology on A.
   (b) The topology generated by $\mathscr{B}_A$ is the subspace topology. Indeed, a subset of A is open in the subspace topology if and only if it is of the form $A \cap V$ with V open in X, that is, of the form $A \cap (\bigcup_{i \in I} B_i)$, with $B_i \in \mathscr{B}$, or, equivalently, of the form $\bigcup_{i \in I}(A \cap B_i)$, that is, a union of elements of $\mathscr{B}_A$.

5. (a) We check the axioms: first any point in Γ clearly lies in some element of $\mathscr{B}$. Second the intersections of two elements of $\mathscr{B}$ come in two flavours: either one gets the intersection of two ε-balls in $\{(x, y) \in \mathbb{R}^2 : y \geq 0\}$ (this also happens when an ε-ball intersects a basic set $A \cup \{z\}$ or when basic open sets $A \cup \{z\}$, $B \cup \{z'\}$ with $z \neq z'$ intersect) or we have two sets $A \cup \{z\}$ and $A' \cup \{z\}$ where A and A' are two balls tangent to the x-axis at z. In the first case, the intersection is a metric open set and thus certainly a union of ε-balls in $\{(x, y) \in \mathbb{R}^2 : y \geq 0\}$. In particular, such an intersection is a union of elements of $\mathscr{B}$. In the second case, one of the balls contains the other (draw a picture) and so the intersection is itself in $\mathscr{B}$.
   (b) The induced topology on $A = \{(x, y) : y > 0\}$ is the usual metric topology. Indeed, by question 4, a base for A is given by the intersections of A with the elements of $\mathscr{B}$. But these intersections are precisely the ε-balls in A which generate the metric topology.
   (c) Again we get the metric topology and by the same argument: the intersections of the basic open sets with the y-axis are the intervals of the form $[0, \varepsilon)$ and (a, b) with $a > 0$. These are precisely the ε-balls of the induced metric.
   (d) The induced topology on the x-axis is an entirely different kettle of fish but the same argument applies: the only basic open sets that intersect the x-axis are those of the form $\{z\} \cup A$ with z on the x-axis and A an ε-ball tangent to the axis. Such a set intersects the X-axis in the singleton set $\{z\}$. Thus the base for the induced topology consists of all the singleton sets and so the induced topology is the discrete topology.