M55: Exercise sheet 1

1. (a) Let (X, d) be a metric space. Show that if $x \neq y \in X$ then there are metric open sets $U, V \subset X$ with $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

   (b) Conclude that the following topologies are not metric: (i) the indiscrete topology on any set with more an one element; (ii) the Sierpinski 2-point topology; (iii) the Zariski topology on $\mathbb{R}$.

2. Let $X = \{a, b, c\}$ be a set with three elements.

   (a) List all the topologies $\mathscr{T}$ on X for which $\{a\}$ is the only open set with one element.

   (b) List all topologies on X up to homeomorphism.

3. Suppose that $\mathscr{T}_1$ and $\mathscr{T}_2$ are topologies on the same underlying set X.

   (a) Show that $\mathscr{T}_1 \cap \mathscr{T}_2$ is a topology on X.

   (b) Is $\mathscr{T}_1 \cup \mathscr{T}_2$ necessarily a topology on X?

4. (a) Show that the identity map $\mathrm{id} : (X, \mathscr{T}) \to (X, \mathscr{T})$ is continuous.

   (b) Show that any constant map $f : (X, \mathscr{T}) \to (Y, \mathscr{S})$ is continuous.

5. Suppose X is a set with topologies $\mathscr{T}_1$ and $\mathscr{T}_2$. If $\mathscr{T}_1 \subset \mathscr{T}_2$, then we say that $\mathscr{T}_2$ is *finer* than $\mathscr{T}_1$ and that $\mathscr{T}_1$ is *coarser* than $\mathscr{T}_2$. (Thus a finer topology has more open sets.)

   (a) Show that $\mathrm{id} : (X, \mathscr{T}_2) \to (X, \mathscr{T}_1)$ is continuous if and only if $\mathscr{T}_2$ is finer than $\mathscr{T}_1$. Conclude that if $\mathscr{T}_2$ is strictly finer than $\mathscr{T}_1$, then id is a continuous bijection that is not a homeomorphism.

   (b) Show that the usual metric topology on $\mathbb{R}$ is strictly finer than the cofinite topology.

6. Let $\mathscr{T}$ be the Zariski (co-finite) topology on $\mathbb{R}$.

   (a) Let p be a polynomial. Show that $p : (\mathbb{R}, \mathscr{T}) \to (\mathbb{R}, \mathscr{T})$ is continuous.
   Hint: you may find it helpful to note that the composition of two polynomials is a polynomial.

   (b) Show that $\sin : (\mathbb{R}, \mathscr{T}) \to (\mathbb{R}, \mathscr{T})$ is *not* continuous.

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M55: Exercise sheet 1—Solutions

1. (*a*) Let $\varepsilon = d(x, y) > 0$ and set $U = B_{\varepsilon/2}(x)$ and $V = B_{\varepsilon/2}(y)$. Then U, V are certainly open and $x \in U$ and $y \in V$. So the only issue is to see that U and V are disjoint. For this, suppose that $z \in U \cap V$. Then the triangle inequality gives

$$d(x, y) \leq d(x, z) + d(z, y) < \varepsilon/2 + \varepsilon/2 = d(x, y),$$

a palpable contradiction!

(*b*) None of the topologies listed have this property. Indeed, (i) the indiscrete topology has only one non-empty open set; (ii) any two non-empty open subsets in the Zariski topology have finite complement and then their intersection has finite complement (De Morgan!) and so cannot be empty; (iii) If $X = \{a, b\}$ with $\{a\}$ open and $\{b\}$ not, then the only open set in the Sierpinski topology containing b is X which also conatins a.

2. (*a*) These are:

$$\begin{aligned}
\mathscr{T}_1 &= \{\emptyset, \{a\}, X\} \\
\mathscr{T}_2 &= \{\emptyset, \{a\}, \{a, b\}, X\} \\
\mathscr{T}_3 &= \{\emptyset, \{a\}, \{a, c\}, X\} \\
\mathscr{T}_4 &= \{\emptyset, \{a\}, \{b, c\}, X\} \\
\mathscr{T}_5 &= \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, X\}.
\end{aligned}$$

Notice that $\mathscr{T}_2$ and $\mathscr{T}_3$ give homeomorphic spaces. Both have the same singleton set and whether we call the extra element in the set of size two, b or c makes no real difference.

(*b*) A systematic way to organise this is to find first all the topologies with no singleton set, then those with 1, 2 and 3 singleton sets. This gives

$$\begin{aligned}
\mathscr{T}_1 &= \{\emptyset, X\} \\
\mathscr{T}_2 &= \{\emptyset, \{a, b\}, X\} \\
\mathscr{T}_3 &= \{\emptyset, \{a\}, X\} \\
\mathscr{T}_4 &= \{\emptyset, \{a\}, \{a, b\}, X\} \\
\mathscr{T}_5 &= \{\emptyset, \{a\}, \{b, c\}, X\} \\
\mathscr{T}_6 &= \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, X\} \\
\mathscr{T}_7 &= \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\} \\
\mathscr{T}_8 &= \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, X\} \\
\mathscr{T}_9 &= \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}.
\end{aligned}$$

Since we are only interested in topologies up to homeomorphism, it does not matter which point comprises the single set when there is only one of these, or which point fails to comprise a singleton set when there are two of these.
Notice that $\mathscr{T}_4$ differs from $\mathscr{T}_5$ by having the singleton set contained in the set with two elements. Therefore these topologies are genuinely different.

3. (*a*) We check the axioms:
(i) Certainly, $\emptyset, X \in \mathscr{T}_i$, $i = 1, 2$, so that $\emptyset, X \in \mathscr{T}_1 \cap \mathscr{T}_2$.
(ii) If $A, B \in \mathscr{T}_1 \cap \mathscr{T}_2$ then $A, B \in \mathscr{T}_i$, $i = 1, 2$ whence $A \cap B \in \mathscr{T}_i$, $i = 1, 2$ giving $A \cap B \in \mathscr{T}_1 \cap \mathscr{T}_2$.
(iii) If $A_\alpha \in \mathscr{T}_1 \cap \mathscr{T}_2$, $\alpha \in I$, then all A_α are in each T_i, $i = 1, 2$, so that $\bigcup_{\alpha \in I} A_\alpha \in \mathscr{T}_i$, $i = 1, 2$. That is, $\bigcup_{\alpha \in I} A_\alpha \in \mathscr{T}_1 \cap \mathscr{T}_2$.

(*b*) The problem is that the union or intersection of a set in $\mathscr{T}_1$ and a set in $\mathscr{T}_2$ might lie in neither $\mathscr{T}_i$. With that in mind we cast around for a counter-example. Question 2 gives us a good supply of simple topologies to think about and so I take $X = \{a, b, c\}$ and $\mathscr{T}_1 = \{\emptyset, \{a\}, X\}$ and $\mathscr{T}_2 = \{\emptyset, \{b\}, X\}$. Now $\mathscr{T}_1 \cup \mathscr{T}_2 = \{\emptyset, \{a\}, \{b\}, X\}$ which is not a topology since it does not contain $\{a, b\} = \{a\} \cup \{b\}$.

4. (a) Almost nothing to prove here: for $U \in \mathscr{T}$, $\mathrm{id}^{-1}(U) = U \in \mathscr{T}$ as required.

   (b) Let f be constant with value $y \in Y$. Let $V \in \mathscr{S}$. There are two cases: if $y \in V$, then $f^{-1}(V) = X \in \mathscr{T}$; if $y \notin V$ then $f^{-1}(V) = \emptyset \in \mathscr{T}$. Either way, $f^{-1}(V)$ is open so that f is continuous.

5. (a) For any subset $U \subset X$, $\mathrm{id}^{-1}(U) = U$. Thus $\mathrm{id} : (X, \mathscr{T}_2) \to (X, \mathscr{T}_1)$ is continuous if and only if whenever $U \in \mathscr{T}_1$, $\mathrm{id}^{-1}(U) = U \in \mathscr{T}_2$. That is, if and only if $\mathscr{T}_1 \subset T_2$.
   Thus, if $\mathscr{T}_2$ is strictly finer than $\mathscr{T}_1$, $\mathrm{id} : (X, \mathscr{T}_2) \to (X, \mathscr{T}_1)$ is a continuous bijection whose inverse (also id) is not continuous $(X, \mathscr{T}_1) \to (X, \mathscr{T}_2)$.

   (b) First note that any Zariski open set is metric open: if A is Zariski open with complement $x_1 < \cdots < x_n$ then

   $$A = (-\infty, x_1) \cup (x_1, x_2) \cup \cdots \cup (x_n, \infty)$$

   which is certainly metric open. Thus the metric topology is finer than the Zariski topology. It is strictly finer because the interval $(0, 1)$, for example, is metric open but not Zariski open: its complement is infinite!

6. (a) Let $A \subset \mathbb{R}$ be a non-empty Zariski open set with complement $x_1, \ldots, x_n$. Let $q(x) = (x - x_1) \ldots (x - x_n)$ so that q is a polynomial with $A = q^{-1}(\mathbb{R} \setminus \{0\})$. Then $p^{-1}(A) = p^{-1}(q^{-1}(\mathbb{R} \setminus \{0\})) = (q \circ p)^{-1}(\mathbb{R} \setminus \{0\})$ with complement $(q \circ p)^{-1}(\{0\})$. Since $q \circ p$ is polynomial, this last is either finite or all of $\mathbb{R}$ so that $p^{-1}(A)$ is Zariski open.

   (b) Note that $\mathbb{R} \setminus \{0\}$ is Zariski open while $\sin^{-1}(\mathbb{R} \setminus \{0\}) = \mathbb{R} \setminus \pi\mathbb{Z}$ which is not Zariski open (it has infinite complement).