M55: Exercise sheet 0 (revision of set theory)

Let X, Y be sets. I hope the following concepts and notations are familiar:

(a) The *power set* $\mathscr{P}(X)$ of X is the collection of all subsets of X. For example, if $X = \{a, b\}$ then $\mathscr{P}(X) = \{\emptyset, \{a\}, \{b\}, X\}$.

(b) If $A \subset X$ then $X \setminus A = \{x \in X : x \notin A\}$. This is the *complement* of A in X.

(c) Let $f : X \to Y$ be a map. If $B \subset Y$, then the *pre-image* or *inverse image* of B under f is given by:

$$f^{-1}(B) = \{x \in X : f(x) \in B\} \subset X.$$

(d) Less useful, although easier to think about, is the *image* of $A \subset X$ under f given by:

$$f(A) = \{f(x) : x \in A\} \subset Y.$$

Exercises

1. Let $f : X \to Y$ be a map from X to Y. Let $U, V \subset X$ and $A, B \subset Y$. Prove or give counterexamples to the following assertions.
   (a) $f(U \cap V) = f(U) \cap f(V)$;
   (b) $f(U \cup V) = f(U) \cup f(V)$;
   (c) $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$;
   (d) $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$;
   (e) $f(X \setminus U) = Y \setminus f(U)$;
   (f) $f^{-1}(Y \setminus B) = X \setminus f^{-1}(B)$.

   In the light of this, which is better behaved: the image or pre-image?

2. Let $(A_i)_{i \in I}$ be a family of subsets of X. Prove the De Morgan Laws:

$$X \setminus \bigcup_{i \in I} A_i = \bigcap_{i \in I} (X \setminus A_i)$$

$$X \setminus \bigcap_{i \in I} A_i = \bigcup_{i \in I} (X \setminus A_i).$$

3. Suppose $f : X \to Y$ is a bijection with inverse $f^{-1} : Y \to X$.
   (a) Let $B \subset Y$. Show that the pre-image of B under f is equal to the image of B under f^{-1}. (So there is no ambiguity in using $f^{-1}(B)$ for both).
   (b) Let $A \subset X$. Show that the pre-image of A under f^{-1} is the same as the image of f. That is $(f^{-1})^{-1}(A) = f(A)$.

4. Let $\mathscr{A}$ be a collection of subsets of X. Show that the following are equivalent:
   (a) U is a union of sets from $\mathscr{A}$.
   (b) For every $x \in U$ there exists a set $V_x \in \mathscr{A}$ such that $x \in V_x \subset U$.

5. Contemplate the following assertions about a map $f : X \to Y$.
   (a) $f^{-1}(f(U)) = U$ for all $U \subset X$.
   (b) $f(f^{-1}(V)) = V$ for all $V \subset Y$.

   Give counter-examples to show that neither is true in general.
   Under what conditions on f is one or other of these assertions true?

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M55: Exercise sheet 0—Solutions

1. When proving that two sets A and B are equal, we usually follow the time-honoured strategy of first showing that $A \subset B$ and then showing that $B \subset A$.
   If, on the other hand, the assertion seems false, we cast around for counter-examples involving *small* sets.

   (a) This is false: suppose that f is not injective and let $a \neq b \in X$ such that $f(a) = f(b) = c \in Y$. Let $U = \{a\}$ and $V = \{b\}$. Then

   $$f(U \cap V) = f(\emptyset) = \emptyset \neq f(U) \cap f(V) = \{c\}.$$

   This is the only thing that can go wrong: if f is injective then the assertion is true.

   (b) This is true. First note that $f(U), f(V)$ are both subsets of $f(U \cup V)$ so that their union is also contained in $f(U \cup V)$. For the converse, if $y \in f(U \cup V)$ then $y = f(x)$ for some $x \in U \cup V$. So $x \in U$ or $x \in V$. That is $f(x)$ is in $f(U)$ or $f(V)$. In other words $y = f(x) \in f(U) \cup f(V)$.

   (c) This is true: if $x \in f^{-1}(A \cap B)$ then $f(x) \in A \cap B$. Thus $f(x) \in A$ and $f(x) \in B$. That is, $x \in f^{-1}(A)$ and $x \in f^{-1}(B)$. Otherwise said, $x \in f^{-1}(A) \cap f^{-1}(B)$ whence $f^{-1}(A \cap B) \subset f^{-1}(A) \cap f^{-1}(B)$.
   Conversely, if $x \in f^{-1}(A) \cap f^{-1}(B)$ then $x \in f^{-1}(A)$ and $x \in f^{-1}(B)$. That is, $f(x) \in A$ and $f(x) \in B$ so that $f(x) \in A \cap B$. This means $x \in f^{-1}(A \cap B)$ so that $f^{-1}(A) \cap f^{-1}(B) \subset f^{-1}(A \cap B)$.

   (d) This is also true: the same argument works replacing all occurrences of $\cap$ by $\cup$ and "and" by "or". To spell it out: if $x \in f^{-1}(A \cup B)$ then $f(x) \in A \cup B$. Thus $f(x) \in A$ or $f(x) \in B$. That is, $x \in f^{-1}(A)$ or $x \in f^{-1}(B)$. Otherwise said, $x \in f^{-1}(A) \cup f^{-1}(B)$ whence $f^{-1}(A \cup B) \subset f^{-1}(A) \cup f^{-1}(B)$.
   Conversely, if $x \in f^{-1}(A) \cup f^{-1}(B)$ then $x \in f^{-1}(A)$ or $x \in f^{-1}(B)$. That is, $f(x) \in A$ or $f(x) \in B$ so that $f(x) \in A \cup B$. This means $x \in f^{-1}(A \cup B)$ so that $f^{-1}(A) \cup f^{-1}(B) \subset f^{-1}(A \cup B)$.

   (e) This is false: for f not surjective we can take $U = \emptyset$ and then $f(X \setminus U) = f(X) \neq Y = Y \setminus f(U)$. Even if f is surjective, things can still go wrong: suppose f is surjective but not injective and suppose a and b are distinct elements that map both to c. Let $U = \{a\}$. Then

   $$f(X \setminus U) = f(X \setminus \{a\}) = f(X) \neq f(X) \setminus \{c\} = Y \setminus \{c\} = Y \setminus f(U).$$

   If, however, f is bijective there are no problems.

   (f) This is true: $x \in f^{-1}(Y \setminus B)$ if and only if $f(x) \notin B$ if and only if $x \notin f^{-1}(B)$ if and only if $x \in X \setminus f^{-1}(B)$.

   The punch-line is that the map $U \mapsto f(U)$ from $\mathscr{P}(X)$ to $\mathscr{P}(Y)$ does not respect the operations of set theory while the "pre-image map" $A \mapsto f^{-1}(A)$ from $\mathscr{P}(Y)$ to $\mathscr{P}(X)$ commutes with everything and so is an altogether nicer operation.

2. For the first law: $x \in X \setminus \bigcup_{i \in I} A_i$ if and only if $x \notin \bigcup_{i \in I} A_i$ if and only if, for every $i \in I$, $x \notin A_i$ if and only if, for every $i \in I$, $x \in X \setminus A_i$ if and only if $x \in \bigcap_{i \in I}(X \setminus A_i)$.
   For the second, $x \in X \setminus \bigcap_{i \in I} A_i$ if and only if $x \notin \bigcap_{i \in I} A_i$ if and only if, for some i, $x \notin A_i$ if and only if, for some i, $x \in X \setminus A_i$ if and only if $x \in \bigcup_{i \in I}(X \setminus A_i)$.

3. (a) The image of B under f^{-1} is the set $\{f^{-1}(y) : y \in B\}$. But this is equal to $\{x \in X : y = f(x) \in B\}$ which is the preimage of B under f.

   (b) The preimage of A under f^{-1} is the set $\{y \in Y : f^{-1}(y) \in A\}$. But this is equal to $\{y \in Y : y = f(x) \text{ for some } x \in A\}$. The last set is the image of A under f.

4. First we show (a) implies (b): let $x \in U$. Now U is a union of sets from $\mathscr{A}$ so that x is contained in some V_x from this union.

Now for the converse. For each $x \in U$, choose some $V_x \in \mathscr{A}$ such that $x \in V_x$ and $V_x \subset U$. Then

$$U = \bigcup_{x \in U} \{x\} \subset \bigcup_{x \in U} V_x \subset U.$$

We must therefore have equality here so that U is a union of sets from $\mathscr{A}$.

5. (a) This is true if and only if f is injective: if $a \neq b \in X$ with $f(a) = f(b) = c$, then, with $U = \{a\}$, we have $f(U) = \{c\}$ and so $\{a, b\} \subset f^{-1}(f(U))$. Thus $f^{-1}(f(U)) \neq U$.
On the other hand, we clearly have $U \subset f^{-1}(f(U))$ and, if f is injective and $x \in f^{-1}(f(U))$ then $f(x) \in f(U)$ so that, for some $u \in U$, $f(x) = f(u)$. Now injectivity gives $x = u$ so that $x \in U$ and $f^{-1}(f(U)) \subset U$.

(b) This is true if and only if f is surjective: if $y \notin f(X)$ then, with $V = \{y\}$, we have $f^{-1}(V) = \emptyset$ so that $f(f^{-1}(V)) = \emptyset \neq V$.
On the other hand, it is clear that $f(f^{-1}(V)) \subset V$ and, if f is surjective and $y \in V$, there is some $x \in X$ with $f(x) = y$ so that $x \in f^{-1}(V)$ giving $y \in f(f^{-1}(V))$.