1. (a) Let X be a topological space and let $x, y \in X$.

What is *a path in* X *from* x *to* y?

Let γ_0, γ_1 be paths in X from x to y. What does it mean to say that γ_0 *is based homotopic to* γ_1?

Show that based homotopy is an equivalence relation. [8]

(b) Give an example of a topological space X and $x, y \in X$ such that

$$\pi_1(X, x) \not\cong \pi_1(X, y).$$

[4]

(c) Denote by D the closed unit disc in $\mathbb{R}^2$ whose boundary is the unit circle S^1.

Show that a continuous map $\phi : S^1 \to X$ into a topological space X is homotopic to a constant map if there is a continuous map $\Phi : D \to X$ such that

$$\Phi|_{S^1} = \phi.$$

[8]

2. (a) What does it mean to say that a topological space is *simply connected*? [2]

(b) Let X be a topological space and $X = U \cup V$ where U, V are simply connected open subsets with $U \cap V$ non-empty and path-connected. Prove that X is simply connected. [10]

(c) Let $x_0 \in X$ a topological space and set $Y = X \setminus \{x_0\}$ equipped with the topology induced from X.

Give examples of x_0 and X satisfying each of the following properties:

(i) Y is simply connected but X is not.

(ii) X is simply connected but Y is not.

(iii) X and Y are both simply connected.

(iv) Neither X nor Y are simply connected.

In each case, briefly justify your answer. [8]

3. (a) A topological space X has the *fixed point property* if every continuous map $f : X \to X$ has a fixed point, that is, there is $x \in X$ such that $f(x) = x$.

   (i) Prove the following version of the Brouwer Fixed Point Theorem: the closed unit disc in $\mathbb{R}^2$ has the fixed point property.
   [You may assume the continuity of any retraction you define.] [8]

   (ii) Which of the following topological spaces has the fixed point property? In each case, briefly justify your answer.

      (1) The unit square $[0, 1] \times [0, 1]$.
      (2) $\mathbb{R}^2$.
      (3) The n-dimensional sphere S^n.

[6]

(b) State the Borsuk–Ulam theorem. [2]
Prove that there is no continuous injection $\mathbb{R}^{n+1} \to \mathbb{R}^n$. [4]

4. (a) Let $p : E \to X$ be a continuous surjective map of topological spaces.
What does it mean to say that p is a *covering map*? [2]
State and prove the *Unique Lifting Property* of covering maps. [8]

(b) Let $p : E \to X$ be a covering map with X path-connected.
For $x, y \in X$, prove that there is a bijection from $p^{-1}(\{x\})$ to $p^{-1}(\{y\})$. [10]