1. (a) Let X be a topological space and $\gamma_0, \gamma_1 : [0,1] \to X$ be paths in X.

(i) If $\gamma_0(1) = \gamma_1(0)$, what is the *join* $\gamma_0 \cdot \gamma_1$ of γ_0 and γ_1? [2]

(ii) What does it mean to say that γ_0 is *based homotopic* to γ_1, that is, $\gamma_0 \sim \gamma_1$? [2]

(iii) For a path $\gamma : [0,1] \to X$, define $\overline{\gamma} : [0,1] \to X$ by $\overline{\gamma}(t) = \gamma(1-t)$, for $t \in [0,1]$. If $\gamma_0 \sim \gamma_1$, show that $\overline{\gamma_0} \sim \overline{\gamma_1}$. [2]

(iv) For a path $\gamma : [0,1] \to X$, show that $\gamma \cdot \overline{\gamma}$ is based homotopic to a constant path. [4]

(b) What is a *retract* of a topological space? [2]

What is a *simply connected* topological space? [2]

Show that a retract of a simply connected topological space is simply connected. [6]

2. Let X, Y, Z be topological spaces.

(a) (i) Let $\phi_0, \phi_1 : X \to Y$ be continuous maps.
What does it mean to say that ϕ_0 and ϕ_1 are *homotopic*: $\phi_0 \simeq \phi_1$? [2]

(ii) Show that homotopy is an equivalence relation on the set of continuous maps $X \to Y$. [5]

(iii) Let $\phi_0, \phi_1 : X \to Y$ and $\psi_0, \psi_1 : Y \to Z$ be continuous maps with $\phi_0 \simeq \phi_1$ and $\psi_0 \simeq \psi_1$.
Show that $\psi_0 \circ \phi_0 \simeq \psi_1 \circ \phi_1$. [3]

(b) (i) Let $f : X \to Y$ be a continuous map.
What does it mean to say that f is a *homotopy equivalence*? [2]

(ii) Let $f_0, f_1 : X \to Y$ be continuous and suppose that f_0 is a homotopy equivalence and that $f_0 \simeq f_1$.
Show that f_1 is also a homotopy equivalence. [4]

(iii) Let $f : X \to Y$ and $g, h : Y \to X$ be continuous maps such that

$$g \circ f \simeq \mathrm{id}_X, \qquad f \circ h \simeq \mathrm{id}_Y.$$

Show that f is a homotopy equivalence. [4]

3. (a) State and prove the Fundamental Theorem of Algebra. [13]

(b) Prove that, when $n > 2$, the 2-sphere S^2 is not homeomorphic to the n-sphere S^n. [7]

4. (a) State the Borsuk–Ulam theorem and prove it for the case of maps $S^2 \to S^1$. [14]

(b) Use the Borsuk–Ulam theorem to prove that there exists no continuous, injective map $\mathbb{R}^{n+1} \to \mathbb{R}^n$. [6]