University of Bath

DEPARTMENT OF MATHEMATICAL SCIENCES

MA40040

Algebraic topology

Wednesday 20th January 2016
09:30 – 11:30
2 hours

Full marks will be given for correct answers to THREE questions.
Only the best three answers will contribute towards the assessment.

No calculators may be brought in and used.

PLEASE FILL IN THE DETAILS ON THE FRONT OF YOUR ANSWER BOOK/COVER AND SIGN IN THE SECTION ON THE RIGHT OF YOUR ANSWER BOOK/COVER, PEEL AWAY ADHESIVE STRIP AND SEAL.

TAKE CARE TO ENTER THE CORRECT CANDIDATE NUMBER AS DETAILED ON YOUR DESK LABEL.

DO NOT TURN OVER YOUR QUESTION PAPER UNTIL INSTRUCTED TO BY THE CHIEF INVIGILATOR.

1. (a) Let X be a topological space and $\gamma_0, \gamma_1, \gamma_2 : [0,1] \to X$ be paths in X.

   (i) If $\gamma_0(1) = \gamma_1(0)$, what is the *join* $\gamma_0 \cdot \gamma_1$ of γ_0 and γ_1? [2]

   (ii) What does it mean to say that γ_0 is *based homotopic* to γ_1, that is, $\gamma_0 \sim \gamma_1$? [2]

   (iii) If $\gamma_0(1) = \gamma_1(0)$ and $\gamma_1(1) = \gamma_2(0)$, prove carefully that

$$(\gamma_0 \cdot \gamma_1) \cdot \gamma_2 \sim \gamma_0 \cdot (\gamma_1 \cdot \gamma_2).$$

[8]

   (b) What are the fundamental groups of the following spaces? Give a brief justification of your answers.

   (i) $\mathbb{R}^3 \setminus \{0\}$.

   (ii) $\{(x,y,z) \in \mathbb{R}^3 \colon |x| + |y| + |z| \leq 1\}$.

   (iii) $S^1 \times S^1 \times S^1$.

[8]

2. (a) (i) What does it mean to say that two topological spaces are *homotopy equivalent*: $X \simeq Y$?
   What does it mean to say that a topological space is *contractible*? [4]

   (ii) Show that a contractible space is path-connected. [4]

   (iii) What is a *retract* of a topological space?
   Show that a retract of a contractible space is contractible. [6]

   (b) (i) State the Brouwer Fixed Point Theorem. [2]

   (ii) Let $A \subset D^2$ be a retract of the closed unit disc. Show that any continuous map $\phi : A \to A$ has a fixed point. [4]

3. (a) What does it mean to say that a topological space X is *simply connected*? [2]

   (b) Let $X = U \cup V$ be a topological space with open subsets U, V such that U, V, and $U \cap V$ are path-connected, U is simply connected and $U \cap V \neq \emptyset$.
   Let $\iota : V \to X$ be the inclusion map and $x_0 \in U \cap V$. Show that $\iota_* : \pi_1(V, x_0) \to \pi_1(X, x_0)$ is surjective. [14]

   (c) Deduce that the n-sphere S^n is simply connected when $n \geq 2$.
   (You may assume that $S^n \setminus \{x\} \cong \mathbb{R}^n$, for any $x \in S^n$.) [4]

4. (a) (i) State the Borsuk–Ulam theorem. [2]

(ii) Let $f : S^2 \to \mathbb{R}^2$ be a continuous map. Show that there is $x_0 \in S^2$ such that $f(x_0) = f(-x_0)$. [4]

(iii) Let $g : S^2 \to S^2$ be a continuous injective map. Show that g is surjective. [4]

(b) (i) State the Ultimate Lifting Theorem. [2]

(ii) Prove that any continuous map from $\mathbb{R}P^2$ to S^1 is homotopic to a constant. [8]

[**Hint**: Show that any such map has a lift $\mathbb{R}P^2 \to \mathbb{R}$.]