1. (a) Let X be a topological space and $\gamma_0, \gamma_1 : [0,1] \to X$ be paths in X.

   (i) If $\gamma_0(1) = \gamma_1(0)$, what is the *join* $\gamma_0 \cdot \gamma_1$ of γ_0 and γ_1? [2]

   (ii) What does it mean to say that γ_0 is *based homotopic* to γ_1, thus $\gamma_0 \sim \gamma_1$. [2]

   (iii) Suppose that γ_0 is a constant path and that $\gamma_0(1) = \gamma_1(0)$. Prove carefully that $\gamma_0 \cdot \gamma_1 \sim \gamma_1$. [4]

   (b) Let $x, y \in X$ where X is a path-connected topological space. Prove that $\pi_1(X,x) \cong \pi_1(X,y)$. [4]

   Show, by a counter-example, that this result is false when X is not path-connected. [2]

   (c) Let A be a subset of a topological space X. What does it mean to say that A is a *retract* of X? [2]

   Show that S^1 is not a retract of $\mathbb{R}^2$. [4]

2. (a) Let $\phi, \psi : X \to Y$ be continuous maps of topological spaces.

   What does it mean to say that ϕ and ψ are *homotopic*, thus $\phi \simeq \psi$? [2]

   Prove that homotopy is an equivalence relation. [4]

   (b) What does it mean to say that two topological spaces are *homotopy equivalent*? [2]

   What does it mean to say that a topological space is *contractible*? [2]

   Show that X is contractible if and only if the identity map $\mathrm{id} : X \to X$ is homotopic to a constant map. [5]

   (c) A subset A of $\mathbb{R}^n$ is *star-convex* if there is $a_0 \in A$ such that all the line segments joining a_0 to other points of A lie in A.

   Show that a star-convex set is contractible. [5]

3. (a) Let $\phi : (X, x) \to (Y, y)$ be a continuous map of topological spaces.

Explain, with proofs, how ϕ defines a homomorphism $\phi_* : \pi_1(X, x) \to \pi_1(Y, y)$. [6]

Now let $\psi : (X, x) \to (Y, y')$ be a continuous map with $\phi \simeq \psi$. Describe, without proof, the relationship between ϕ_* and ψ_*. [2]

(b) Let $A \subset \mathbb{R}^n$ and $h : (A, a) \to (Y, y)$ be a continuous map to a topological space Y.

Suppose that there is a continuous map $H : \mathbb{R}^n \to Y$ such that $H_{|A} = h$. Show that $h_* : \pi_1(A, a) \to \pi_1(Y, y)$ is constant. [6]

(c) Let $\phi : S^1 \to S^1$ be continuous and without fixed points: $\phi(x) \neq x$, for all $x \in S^1$.

Show that ϕ is homotopic to the identity map of S^1. [6]

[**Hint**: start by showing that ϕ is homotopic to $-\mathrm{id}$.]

4. (a) Let $p : E \to X$ be a continuous surjection of topological spaces.

What does it mean to say that p is a *covering map*? [2]

(b) What is a *universal cover* of a topological space? [2]

What does it mean to say that a topological space is *semi-locally simply connected*? [2]

Prove that if a topological space has a universal cover then it is semi-locally simply connected. [4]

(c) Let $p : (E, e_0) \to (X, x_0)$ be a covering map. Show that the induced map $p_* : \pi_1(E, e_0) \to \pi_1(X, x_0)$ is injective. [4]

[You may state without proof any lifting results that you need.]

(d) Let $p : E \to X$ be a covering map with E path-connected and X simply connected.

Show that p is a homeomorphism. [6]

[You may assume that covering maps are open.]