1. Let X be a topological space.

(a) Let $x \in X$. What is the *fundamental group* $\pi_1(X, x)$ *of* X *based at* x?
[You should carefully describe the elements, group law, neutral element and inverses of $\pi_1(X, x)$ but you need not prove any assertions you make.]

(b) Let $\phi : X \to Y$ be a continuous map of topological spaces with X path-connected. Show that $\phi(X)$ is path-connected.

(c) Let $X_0 \subset X$ be a path component of X and $\iota : X_0 \to X$ the inclusion.
For $x \in X_0$, show that the induced homomorphism $\iota_* : \pi_1(X_0, x) \to \pi_1(X, x)$ is an isomorphism.

(d) Give an example of a topological space X and $x_0, x_1 \in X$ such that $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$ are *not* isomorphic.

(e) Show that X is path-connected if and only if, for any topological space Y, any two *constant* maps $\phi_0, \phi_1 : Y \to X$ are homotopic.

2. (a) Let $\phi_0, \phi_1 : X \to Y$ be continuous maps of topological spaces. What does it mean to say that ϕ_0 *is homotopic to* ϕ_1: $\phi_0 \simeq \phi_1$?
Now let $\psi_0, \psi_1 : Y \to Z$ also be continuous maps with $\phi_0 \simeq \phi_1$ and $\psi_0 \simeq \psi_1$. Show that

$$\psi_0 \circ \phi_0 \simeq \psi_1 \circ \phi_1.$$

(b) What does it mean to say that two topological spaces are *homotopy equivalent*?

(c) Show that homotopy equivalence is an equivalence relation on topological spaces.

(d) Let $X \simeq Y$ via a homotopy equivalence $f : X \to Y$ and let $x \in X$. Does it follow that f induces a homotopy equivalence $X \setminus \{x\} \to Y \setminus \{f(x)\}$?
[Justify your answer with a proof or counter-example.]

(e) Let S^n denote the unit sphere in $\mathbb{R}^{n+1}$ and let X be defined by:

$$X = \{(p, q) \in S^n \times S^n : p \neq -q\}.$$

Show that $S^n \simeq X$.

3. (a) What is a *covering map*?

(b) Use the covering map $\phi : \mathbb{R} \to S^1$ given by $\phi(t) = e^{2\pi it}$ to prove that $\pi_1(S^1, 1) \cong \mathbb{Z}$. [You may state without proof any lifting results that you need.]

(c) Let $\Phi : \pi_1(S^1, 1) \to \pi_1(S^1, 1)$ be a group homomorphism. Show that Φ is induced by a continuous map $\phi : S^1 \to S^1$.

4. (a) State the Ultimate Lifting Theorem.

(b) Show that the quotient map $p : S^2 \to \mathbb{R}P^2$ is a covering map.

(c) Let $\phi : \mathbb{R}P^2 \to \mathbb{R}P^2$ be a continuous map such that the induced homomorphism $\phi_* : \pi_1(\mathbb{R}P^2) \to \pi_1(\mathbb{R}P^2)$ is not constant.
Show that there is a continuous map $\psi : S^2 \to S^2$ such that

1. $p \circ \psi = \phi \circ p$;
2. $\psi(-x) = -\psi(x)$ for all $x \in S^2$.

Hint: Use the Ultimate Lifting Theorem and argue by contradiction.