

## M40: Christmas exercise sheet

### Some revision questions

1. (1994 exam) Let  $n \in \mathbb{N}$ . The map  $p : S^1 \rightarrow S^1$  given by  $p(e^{it}) = e^{int}$  is a covering map with group of deck translations  $H$ . Show that

$$H \cong \mathbb{Z}/n\mathbb{Z}.$$

2. (1991 exam) Let  $f : S^1 \rightarrow S^1$  be continuous and not homotopic to the identity map. Show that, for some  $x \in S^1$ ,  $f(x) = -x$
3. (1992 exam) Let  $X$  be a topological space and  $f : X \rightarrow S^n$  be a continuous map. If  $f$  is not surjective, show that  $f$  is homotopic to a constant.

### More substantial questions

4. Let  $X, Y$  and  $Z$  be topological spaces with  $Y$  strongly locally compact. Show that the map  $c : \mathcal{C}(X, Y) \times \mathcal{C}(Y, Z) \rightarrow \mathcal{C}(X, Z)$  given by

$$c(f, g) = g \circ f$$

is continuous.

5. First a little group theory: let  $G$  be a group with subgroup  $H$ . The *normaliser of  $H$  in  $G$* , denoted  $N_G(H)$ , is given by

$$N_G(H) = \{g \in G : gHg^{-1} = H\}.$$

It is easy to show:

- (a)  $N_G(H)$  is a subgroup of  $G$  containing  $H$ ;
- (b)  $H$  is a normal subgroup of  $N_G(H)$  (so that the coset space  $N_G(H)/H$  is a group).

Back to topology: let  $p : (E, e_0) \rightarrow (X, x_0)$  be a covering map with  $E$  connected and locally path-connected but not necessarily simply connected. Recall that  $p_* : \pi_1(E, e_0) \rightarrow \pi_1(X, x_0)$  is an injection. Set  $G = \pi_1(X, x_0)$  and let  $H = p_*(\pi_1(E, e_0)) \subset G$ .

Finally let  $\mathcal{G}$  be the group of deck translations of  $E$ . Show that

$$\mathcal{G} \cong N_G(H)/H.$$

(Observe that this reduces to Theorem 3.8 when  $E$  is simply-connected.)

6. Let  $G$  be a topological group with a simply connected, locally path-connected covering space  $p : \tilde{G} \rightarrow G$ . Choose  $e \in p^{-1}\{1\}$ , the fibre over the identity of  $G$ . Show that  $\tilde{G}$  can be equipped with the structure of a topological group in such a way that  $e$  is the identity element and  $p$  is a homomorphism.

**Hint:** Use the Ultimate Lifting Theorem to define the multiplication and inversion maps and make repeated use of the Unique Lifting Property to establish the group axioms.

7. An  $H$ -space is a topological space  $\Omega$  with two extra ingredients:
  - (i) A continuous map  $m : \Omega \times \Omega \rightarrow \Omega$  written  $m(x, y) = x * y$ ;
  - (ii) A *homotopy identity* for  $m$ , that is, an element  $e \in \Omega$  such that the maps  $L, R : \Omega \rightarrow \Omega$  given by  $L(x) = e * x, R(x) = x * e$  are both homotopic to the identity relative to  $\{e\}$ . (In particular,  $e * e = e$ .)

The remaining exercises explore examples and properties of certain  $H$ -spaces.

(a) Show that any topological group is an  $H$ -space.

Now let  $\Omega$  be an  $H$ -space and let  $\gamma_1, \gamma_2$  and  $\gamma_3$  be loops in  $\Omega$  based at  $e$ . Define a new loop  $\gamma_1 * \gamma_2$  by

$$\gamma_1 * \gamma_2(t) = \gamma_1(t) * \gamma_2(t).$$

(b) Show that if  $\gamma_2 \sim \gamma_3$ , then  $\gamma_1 * \gamma_2 \sim \gamma_1 * \gamma_3$ .

(c) Show that  $\gamma_1 \sim \gamma_1 * \gamma_e \sim \gamma_e * \gamma_1$  where, as usual  $\gamma_e$  is the constant loop based at  $e$ .

(d) Deduce that  $\gamma_1 \cdot \gamma_2 \sim \gamma_1 * \gamma_2 \sim \gamma_2 \cdot \gamma_1$  and hence that  $\pi_1(\Omega, e)$  is abelian.

In particular, if  $G$  is a topological group,  $\pi_1(G, 1)$  is abelian.

8. Let  $X$  be a topological space and fix  $x_0 \in X$ . Let  $\Omega X$  be the subspace of  $\mathcal{C}(I, X)$  the space of continuous maps  $I \rightarrow X$  given by

$$\Omega X = \{\alpha \in \mathcal{C}(I, X) : \alpha(0) = \alpha(1) = x_0\}$$

Thus  $\Omega X$  is the space of loops based at  $x_0$  that we previously called  $\text{Path}(X, x_0, x_0)$ .

We give  $\Omega X$  the topology induced from the compact open topology on  $\mathcal{C}(I, X)$ .

(a) Show that  $m : \Omega X \times \Omega X \rightarrow \Omega X$  given by

$$m(\alpha_0, \alpha_1) = \alpha_0 \cdot \alpha_1$$

is continuous.

(b) Let  $e = \gamma_{x_0} \in \Omega X$  be the constant loop at  $x_0$ . Show that  $e$  is a homotopy identity for  $m$  and thus that  $\Omega X$  is an  $H$ -space.

**Hint:** You may find the result of question 4 useful here.

Since  $I$  is strongly locally compact (why?), we learn from a previous exercise sheet that, for  $\alpha_0, \alpha_1 \in \Omega X$ ,  $\alpha_0 \sim \alpha_1$  if and only if there is a path in  $\Omega X$  from  $\alpha_0$  to  $\alpha_1$ . Thus we have a bijection between  $\pi_0(\Omega X)$  and  $\pi_1(X, x_0)$ . This motivates the

**Definition.** The *second homotopy group* of  $X$  based at  $x_0$ , denoted  $\pi_2(X, x_0)$  is the group  $\pi_1(\Omega X, e)$ .

The  *$n$ -th homotopy group* of  $X$  based at  $x_0$  is then defined inductively by  $\pi_n(X, x_0) = \pi_{n-1}(\Omega X, e)$ .

We deduce from question 7 that  $\pi_n(X, x_0)$  is abelian for all  $n > 1$ .

(c) Show that if  $f : (X, x_0) \rightarrow (Y, y_0)$  is a continuous map then the map  $\Omega(f) : \Omega X \rightarrow \Omega Y$  given by  $\Omega(f)(\alpha_1) = f \circ \alpha_1$  is also continuous.

(d) Use this to define a homomorphism  $f_* : \pi_2(X, x_0) \rightarrow \pi_2(Y, y_0)$  and show that if  $g : (Y, y_0) \rightarrow (Z, z_0)$  is also continuous then

$$(g \circ f)_* = g_* \circ f_*.$$

Finally show that  $\text{id}_*$  is the identity map of  $\pi_2$  so that we have defined another functor!

These properties extend by induction to all  $\pi_n$ .

9. The groups  $\pi_n(X)$  are one of the main families of topological invariants and their study is the vast theory of *Homotopy Theory*. As a final exercise, go to the Library and try and find out about the other main family: the *homology groups*.

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## M40: Christmas exercise sheet—Solutions

- For  $k \in \mathbb{Z}$ , define  $\phi_k : S^1 \rightarrow S^1$  by  $\phi_k(z) = e^{2k\pi i/n}z$ . Certainly these are homeomorphisms with  $\phi_k \circ \phi_l = \phi_{k+l}$  and  $p(\phi_k(e^{it})) = e^{2k\pi i}e^{int} = p(e^{it})$  so that each  $\phi_k$  is a deck translation. Thus we have a homomorphism  $k \mapsto \phi_k : \mathbb{Z} \rightarrow H$  and it is easy to check that its kernel is  $n\mathbb{Z}$ :  $\phi_k = \text{id}_{S^1}$  if and only if  $k \in n\mathbb{Z}$ . Thus we will be done (via the first isomorphism theorem) if we can see that our homomorphism is onto. But if  $\phi \in H$  then  $\phi(1) \in p^{-1}\{1\} = \{e^{2k\pi i/n} : k = 1, \dots, n\}$  so that  $\phi(1) = e^{2k\pi i/n} = \phi_k(1)$ , for some  $k$ . Now both  $\phi$  and  $\phi_k$  are lifts of  $p$  so that the unique lifting property shows us that  $\phi = \phi_k$  and we are done.
- Suppose that, for all  $x \in S^1$ ,  $f(x) \neq -x$ . We shall show that  $f$  is homotopic to the identity map. Indeed, define  $F : I \times S^1 \rightarrow S^1$  by

$$F(t, x) = \frac{tf(x) + (1-t)x}{\|tf(x) + (1-t)x\|}.$$

$F$  is continuous as long as it is well-defined and it is well-defined since  $tf(x) + (1-t)x = 0$  for some  $x \in S^1$  and  $t \in I$  if and only if  $f(x) = -x$ . It is now straight-forward to see that  $F$  is the desired homotopy.

- Conceptual answer: if  $f$  is not surjective then  $f$  has image in  $S^n \setminus \{x_0\}$ , for some  $x_0$ , and this is homeomorphic to  $\mathbb{R}^n$  which is contractible. Any continuous map into a contractible space is easily seen to be homotopic to a constant (why?).  
Low-tech answer: with  $x_0 \notin f(X)$  as above, we show that  $f$  is homotopic to be constant map with value  $-x_0$ . Indeed, it is easy to check that  $F : I \times S^n \rightarrow S^n$  given by

$$F(t, x) = \frac{tf(x) - (1-t)x_0}{\|tf(x) - (1-t)x_0\|}$$

is well-defined (and hence continuous) and thus provides the desired homotopy.

- Let  $K \subset X$  be compact and  $G \subset Z$  be continuous. Let  $(f, g) \in c^{-1}(\mathcal{C}_{K,G})$  so that  $g \circ f(K) \subset G$ . Thus  $f(K) \subset g^{-1}(G) \subset Y$  and  $f(K)$  is compact and  $g^{-1}(G)$  is open. We would be done if we could find open sets  $f\mathcal{C}_{K,N^\circ} \subset \mathcal{C}(X, Y)$  and  $g \in \mathcal{C}_{N,G} \subset \mathcal{C}(Y, Z)$  with

$$\mathcal{C}_{K,N^\circ} \times \mathcal{C}_{N,G} \subset c^{-1}(\mathcal{C}_{K,G}).$$

(Here  $N^\circ, N \subset Y$  with  $N^\circ$  open and  $N$  compact). For this to have a chance of working we need  $f(K) \subset N^\circ$  (to ensure  $f \in \mathcal{C}_{K,N^\circ}$ ),  $g(N) \subset G$  (to ensure  $g \in \mathcal{C}_{N,G}$  and finally  $N^\circ \subset N$  (to ensure that compositions send  $K$  to  $G$ ). We therefore need to prove the following lemma:

**Lemma 0.1.** *Let  $Y$  be strongly locally compact and  $K \subset G \subset Y$  with  $K$  compact and  $G$  open. Then there is a compact set  $N \subset Y$  such that*

$$K \subset N^\circ \subset N \subset G,$$

where  $N^\circ$  denotes the interior of  $N$ .

*Proof.* Each  $k \in K$  has a compact neighbourhood  $U_k \subset G$ . The interiors of all these comprise an open cover of  $K$  and so passing to a finite subcover gives  $N = \bigcup_{i=1}^n U_{k_i} \subset G$  with  $K \subset N^\circ$ . Moreover,  $N$  is compact being a finite union of compact sets.  $\square$

The lemma now gives us a compact set  $N \subset Y$  with  $f(K) \subset N^\circ \subset N \subset g^{-1}(G)$ . Thus, if  $f' \in \mathcal{C}_{K,N^\circ}$  and  $g' \in \mathcal{C}_{N,G}$ , then  $g' \circ f'(K) \subset G$  and we have proved that

$$(f, g) \in \mathcal{C}_{K,N^\circ} \times \mathcal{C}_{N,G} \subset c^{-1}(\mathcal{C}_{K,G}).$$

It now follows that  $c^{-1}(\mathcal{C}_{K,G})$  is open so that  $c$  is continuous.

5. The assertions about normalisers are utterly straight-forward so let's get down to the nitty-gritty. The idea is to mimic the proof of Theorem 3.8 and define a map from  $\mathcal{G}$  to  $N_G(H)/H$  as follows: let  $\phi \in \mathcal{G}$  and choose a path  $\sigma'$  from  $e_0$  to  $\phi(e_0)$ . Then  $\sigma = p \circ \sigma'$  is a loop at  $x_0$  and hopefully its homotopy class will give us an element of  $N_G(H)/H$ . Let us see how this works out:

First we show that  $[\sigma] \in N_G(H)$ . So let  $[\tau] \in H$ . Then  $[\tau] = [p \circ \tau']$  for some loop  $\tau'$  in  $E$  based at  $e_0$  and we must show that  $[\sigma \cdot p \circ \tau' \cdot \sigma^{-1}] \in H$ . But  $\phi \circ \tau'$  is a loop in  $E$  based at  $\phi(e_0)$  and covering  $p \circ \tau'$  so that  $\sigma' \cdot (\phi \circ \tau') \cdot \sigma'^{-1}$  is a well-defined loop at  $e_0$  covering  $\sigma \cdot p \circ \tau' \cdot \sigma^{-1}$ . This means that  $[\sigma \cdot p \circ \tau' \cdot \sigma^{-1}] \in H$  as required.

So far, the element of  $N_G(H)$  that we have constructed depends on the path  $\sigma'$  from  $e_0$  to  $\phi(e_0)$ . So let  $\rho'$  be another such path with  $\rho = p \circ \rho'$ . Then  $\sigma' \sim (\sigma' \cdot \rho'^{-1}) \cdot \rho'$  so that, setting  $\tau' = \sigma' \cdot \rho'^{-1}$  which is a loop at  $e_0$  we have

$$[\sigma] = [p \circ \tau'][\rho].$$

Now  $[p \circ \tau'] \in H$  so  $H[\sigma] = H[\rho]$  or, since  $H$  is normal in  $N_G(H)$ ,  $[\sigma]H = [\rho]H$ . To conclude then, we have found that the cosets of  $[\sigma]$  and  $[\rho]$  are the same element of  $N_G(H)/H$  and so have a well-defined map  $\chi : \mathcal{G} \rightarrow N_G(H)/H$  given by

$$\chi(\phi) = [p \circ \sigma']H,$$

where  $\sigma'$  is any path from  $e_0$  to  $\phi(e_0)$ .

We now show that  $\chi$  is an isomorphism. First we show that it is a homomorphism: let  $\phi_1, \phi_2 \in \mathcal{G}$  and let  $\sigma'$  and  $\tau'$  be paths from  $e_0$  to  $\phi_1(e_0)$  and  $\phi_2(e_0)$  respectively. Let  $\sigma = p \circ \sigma'$  and  $\tau = p \circ \tau'$ . Now  $\phi_1 \circ \tau'$  is a path from  $\phi_1(e_0)$  to

$$\phi_1(\tau'(1)) = \phi_1(\phi_2(e_0)).$$

Thus  $\sigma' \cdot (\phi_1 \circ \tau')$  is a path from  $e_0$  to  $\phi_1 \circ \phi_2(e_0)$  so that

$$\chi(\phi_1 \circ \phi_2) = [\sigma \cdot (p \circ \phi_1 \circ \tau')]H = [\sigma][\tau]H = \chi(\phi_1)\chi(\phi_2),$$

as required.

As for injectivity of  $\chi$ : let  $\phi \in \ker \chi$ . We must show that  $\phi = \text{id}_E$ . So let  $\sigma'$  be a path from  $e_0$  to  $\phi(e_0)$  with  $\sigma = p \circ \sigma'$ . By hypothesis,  $[\sigma] \in H$  so that  $\sigma \sim p \circ \tau'$  with  $\tau'$  a loop in  $E$  based at  $e_0$ . The homotopy lifting property then gives that  $\sigma' \sim \tau'$  and, in particular,  $\phi(e_0) = \sigma'(1) = \tau'(1) = e_0$  since  $\tau'$  is a loop. Thus  $\phi$  is a lift of  $p$  which agrees with  $\text{id}_E$  at  $e_0$  and so, by the unique lifting property,  $\phi = \text{id}_E$  as required.

Finally we show surjectivity: so let  $[\sigma]H \in N_G(H)$  and let  $e_1 = \sigma'_{e_0}(1)$ . We must find  $\phi \in \mathcal{G}$  for which  $\phi(e_0) = e_1$ . We will have done this if we can get continuous maps  $\phi, \bar{\phi} : E \rightarrow E$  which lift  $p$  such that  $\phi(e_0) = e_1$  and  $\bar{\phi}(e_1) = e_0$ . For then the usual unique lifting argument shows that  $\phi$  and  $\bar{\phi}$  are mutually inverse deck translations. Now the ultimate lifting theorem guarantees the existence of  $\phi$  as soon as we know that  $p_*(\pi_1(E, e_0)) \subset p_*(E, e_1)$  while  $\bar{\phi}$  exists when  $p_*(\pi_1(E, e_1)) \subset p_*(E, e_0)$ . So it remains to show that these two groups are equal.

For this, let  $[\tau] \in p_*(\pi_1(E, e_0))$  so that we have a loop  $\tau'$  based at  $e_0$  with  $p \circ \tau' = \tau$ . Now  $(\sigma'_{e_0})^{-1} \cdot \tau' \cdot \sigma'_{e_0}$  is a loop at  $e_1$  and taking  $p$  of this, we see that  $[\sigma]^{-1}[\tau][\sigma] \in p_*\pi_1(E, e_1)$ . Thus

$$[\sigma]^{-1}p_*\pi_1(E, e_0)[\sigma] \subset p_*\pi_1(E, e_1).$$

But  $[\sigma]$  is in the normaliser of  $p_*\pi_1(E, e_0)$  so that

$$p_*\pi_1(E, e_0) = [\sigma]^{-1}p_*\pi_1(E, e_0)[\sigma] \subset p_*\pi_1(E, e_1).$$

For the converse, let  $[\tau] \in p_*\pi_1(E, e_1)$  with  $\tau = p \circ \tau'$  for some loop  $\tau'$  based at  $e_1$ . Then  $\sigma'_{e_0} \cdot \tau' \cdot (\sigma'_{e_0})^{-1}$  is a loop at  $e_0$  which we take  $p$  of to conclude that  $[\sigma][\tau][\sigma]^{-1} \in p_*\pi_1(E, e_0)$ . Then

$$[\tau] \in [\sigma]^{-1}p_*\pi_1(E, e_0)[\sigma] = p_*\pi_1(E, e_0)$$

and we are done!

6. The first thing to do is define the multiplication  $\tilde{\mu} : \tilde{G} \times \tilde{G} \rightarrow \tilde{G}$  and the inversion  $\tilde{\beta} : \tilde{G} \rightarrow \tilde{G}$ .

We define  $\tilde{\beta}$  to be the unique lift of  $i \circ p : \tilde{G} \rightarrow G$  such that  $\tilde{\beta}(e) = e$ . Such a lift exists by the Ultimate Lifting Theorem since  $\tilde{G}$  is simply connected. Thus

$$\begin{array}{ccc} \tilde{G} & \xrightarrow{\tilde{\beta}} & \tilde{G} \\ p \downarrow & & \downarrow p \\ G & \xrightarrow{i} & G \end{array}$$

commutes.

Similarly, define  $p \times p : \tilde{G} \times \tilde{G} \rightarrow G \times G$  by  $p \times p(x, y) = (p(x), p(y))$  (this is clearly a continuous map) and define  $\tilde{\mu} : \tilde{G} \times \tilde{G} \rightarrow \tilde{G}$  to be the unique lift of  $\mu \circ (p \times p) : \tilde{G} \times \tilde{G} \rightarrow G$  with  $\tilde{\mu}(e, e) = e$ . Again, this lift exists by the Ultimate Lifting Theorem since  $\tilde{G} \times \tilde{G}$  is simply connected so we get a commuting diagram:

$$\begin{array}{ccc} \tilde{G} \times \tilde{G} & \xrightarrow{\tilde{\mu}} & \tilde{G} \\ p \times p \downarrow & & \downarrow p \\ G \times G & \xrightarrow{\mu} & G \end{array}$$

Now we must show that the multiplication and inversion so defined actually satisfy the group axioms. First associativity: define maps  $\tilde{A}_1, \tilde{A}_2 : \tilde{G} \times \tilde{G} \times \tilde{G} \rightarrow \tilde{G}$  by

$$\begin{aligned} \tilde{A}_1(g_1, g_2, g_3) &= \tilde{\mu}(g_1, \tilde{\mu}(g_2, g_3)) = g_1(g_2g_3) \\ \tilde{A}_2(g_1, g_2, g_3) &= \tilde{\mu}(\tilde{\mu}(g_1, g_2), g_3) = (g_1g_2)g_3 \end{aligned}$$

Both  $\tilde{A}_i$  are continuous and  $\tilde{A}_i(e, e, e) = e$ . However,

$$p(\tilde{A}_1(g_1, g_2, g_3)) = p(g_1)(p(g_2)p(g_3)) = (p(g_1)p(g_2))p(g_3) = p(\tilde{A}_2(g_1, g_2, g_3)),$$

by the associativity of the multiplication in  $G$ , so that by the unique lifting property,  $\tilde{A}_1 = \tilde{A}_2$  and  $\tilde{\mu}$  is associative.

As for  $e$  being the identity, define  $L : \tilde{G} \rightarrow \tilde{G}$  by  $L(g) = \tilde{\mu}(e, g) = eg$  and observe that  $p \circ L(g) = \mu(p(e), p(g)) = 1p(g) = p(g)$  while  $L(e) = \tilde{\mu}(e, e) = e$ . Thus both  $L$  and  $\text{id}_{\tilde{G}}$  lift  $p$  and agree at  $e$  so that, by the unique lifting property,  $L = \text{id}_{\tilde{G}}$ . This means  $eg = g$  for all  $g \in \tilde{G}$ . Similarly  $ge = g$  for all  $g \in \tilde{G}$  whence  $e$  is the identity for  $\tilde{\mu}$ .

Finally, consider  $I : \tilde{G} \rightarrow \tilde{G}$  given by  $I(g) = \tilde{\mu}(g, \tilde{\beta}(g))$ . Clearly  $I$  is continuous and  $I(e) = e$ . Moreover,  $p(I(g)) = \mu(p(g), i(p(g))) = p(g)(p(g))^{-1} = 1$ , for all  $g \in \tilde{G}$ . Thus  $I$  takes values in the discrete set  $p^{-1}\{1\}$  and so is constant giving  $I(g) = 1$  or  $g\tilde{\beta}(g) = 1$ . Similarly,  $\tilde{\beta}(g)g = 1$ , for all  $g \in \tilde{G}$  and we conclude that the group axioms are satisfied.

It is straight-forward to show that if a space is Hausdorff so is any covering space thereof so that  $\tilde{G}$  is Hausdorff and therefore we see that  $\tilde{G}$  is a topological group.

Finally, the commuting diagram describing  $\tilde{\mu}$  gives

$$p(g_1g_2) = p(\tilde{\mu}(g_1, g_2)) = \mu(p(g_1), p(g_2)) = p(g_1)p(g_2)$$

so that  $p$  is a homomorphism.

7. (a) This is very easy: let  $m$  be the usual multiplication and  $e$  the identity element. In this case, the maps  $L$  and  $R$  are equal to the identity map.
- (b) Let  $F$  be a homotopy from  $\gamma_2$  to  $\gamma_3$  and define  $G : I \times I \rightarrow \Omega$  by

$$G(t, s) = m(\gamma_1(t), F(t, s)).$$

$G$  is clearly continuous being a composition of continuous maps and moreover,  $G(0, s) = G(1, s) = m(e, e) = e$  while  $G(t, 0) = m(\gamma_1(t), \gamma_2(t)) = (\gamma_1 * \gamma_2)(t)$  and, similarly,  $G(t, 1) = (\gamma_1 * \gamma_3)(t)$ . Thus  $G$  is the desired homotopy.

- (c) Let  $F : I \times \Omega \rightarrow \Omega$  be the homotopy from the identity to  $L$  relative to  $\{e\}$  and define  $G : I \times I \rightarrow \Omega$  by

$$G(t, s) = F(s, \gamma_1(t))$$

which is certainly continuous. Then  $G(t, 0) = F(0, \gamma_1(t)) = \gamma_1(t)$  while  $G(t, 1) = F(1, \gamma_1(t)) = L(\gamma_1(t)) = e * \gamma_1(t) = (\gamma_e * \gamma_1)(t)$ . Moreover  $G(s, 0) = G(s, 1) = F(s, e) = e$  since  $F$  is a homotopy relative to  $\{e\}$ . Thus  $\gamma_1 \sim \gamma_e * \gamma_1$ .

A similar argument establishes the other identity.

- (d) We know that  $\gamma_1 \sim \gamma_1 \cdot \gamma_e$  and  $\gamma_2 \sim \gamma_e \cdot \gamma_2$ . Thus, from the above, we get

$$\gamma_1 * \gamma_2 \sim (\gamma_1 \cdot \gamma_e) * (\gamma_e \cdot \gamma_2).$$

However, we see from the definitions that

$$(\gamma_1 \cdot \gamma_e) * (\gamma_e \cdot \gamma_2) = (\gamma_1 * \gamma_e) \cdot (\gamma_e * \gamma_2),$$

while the above shows that this last is homotopic to  $\gamma_1 \cdot \gamma_2$ . Thus  $\gamma_1 \cdot \gamma_2 \sim \gamma_1 * \gamma_2$ . The other identity is proved in a similar fashion.

8. Recall the the compact open topology is that topology on  $\mathcal{C}(Y, X)$  with sub-base consisting of all sets  $\mathcal{C}_{K,G}$  of the form

$$\mathcal{C}_{K,G} = \{f \in \mathcal{C}(Y, X) : f(K) \subset G\},$$

where  $K \subset Y$  is compact and  $G \subset X$  is open. We denote the induced sub-basic open sets of  $\Omega X$  by  $\Omega_{K,G}$ : thus

$$\Omega_{K,G} = \{\alpha \in \Omega X : \alpha(K) \subset G\}$$

where  $K \subset I$  is compact and  $G \subset X$  is open.

- (a) It suffices to show that the inverse image of any sub-basic open set of  $\Omega X$  by  $m$  is open. So consider  $m^{-1}(\Omega_{K,G})$  with  $K \subset I$  compact and  $G \subset X$  open. Then  $(\alpha_0, \alpha_1) \in m^{-1}(\Omega_{K,G})$  if and only if  $\alpha_0 \cdot \alpha_1(K) \subset G$  if and only if  $\alpha_0(2t) \in G$  for  $t \in K \cap [0, \frac{1}{2}]$  and  $\alpha_1(2t - 1) \in G$  for  $t \in K \cap [\frac{1}{2}, 1]$ . To say this another way, define  $K_1, K_2 \subset I$  by

$$K_1 = \{2t : t \in K \cap [0, \frac{1}{2}]\} \quad K_2 = \{2t - 1 : t \in K \cap [\frac{1}{2}, 1]\}.$$

These are easily seen to be compact subsets of  $I$  and the above remarks show that

$$\Omega_{K_1,G} \times \Omega_{K_2,G} = m^{-1}(\Omega_{K,G})$$

so that  $m^{-1}(\Omega_{K,G})$  is open in  $\Omega X \times \Omega X$  and  $m$  is continuous.

- (b) We must find homotopies relative to  $\{e\}$  between the identity map  $\text{id} : \Omega X \rightarrow \Omega X$  and the maps  $L, R : \Omega X \rightarrow \Omega X$  given by  $L\alpha = e \cdot \alpha$  and  $R\alpha = \alpha \cdot e$ . We shall just give the argument for  $L$ . It is not so difficult to see what the homotopy must be: any  $\alpha$  is homotopic to  $e \cdot \alpha$  and this suggests defining  $F : I \times \Omega X \rightarrow \Omega X$  by

$$F(s, \alpha)(t) = \begin{cases} \alpha(0) = x_0 & \text{for } 0 \leq t \leq s/2 \\ \alpha((2t - s)/(2 - s)) & \text{for } s/2 \leq t \leq 1 \end{cases}$$

It is easy to check that  $F$  is well-defined, that  $F(0, \alpha) = \alpha$ ,  $F(1, \alpha) = e \cdot \alpha$  and  $F(s, e) = e$ , for all  $\alpha \in \Omega X$  and  $s \in I$ . Thus  $F$  is the required homotopy if we can prove that it is continuous. For this, we argue as follows: on a previous sheet we proved that when  $Y$  is strongly locally compact,  $f, g \in \mathcal{C}(Y, Z)$  were homotopic if and only if there was a path in  $\mathcal{C}(Y, Z)$  from  $f$  to  $g$ . This amounted to showing that  $G : I \times Y \rightarrow Z$  is continuous if and only if the map  $\widehat{G} : I \rightarrow \mathcal{C}(Y, Z)$  given by  $\widehat{G}(s)(y) = G(s, y)$  is continuous. We apply this result to the map  $G : I \times I \rightarrow I$  given by

$$G(s, t) = \begin{cases} 0 = x_0 & \text{for } 0 \leq t \leq s/2 \\ (2t - s)/(2 - s) & \text{for } s/2 \leq t \leq 1 \end{cases}$$

which is obviously continuous. The point now is that  $F(s, \alpha) = \alpha \circ \widehat{G}(s)$  and so is the composition of the continuous map  $(s, \alpha) \mapsto (\widehat{G}(s), \alpha) : I \times \Omega X \rightarrow \mathcal{C}(I, I) \times \Omega X$  and the composition map  $c : \mathcal{C}(I, I) \times \mathcal{C}(I, X)$  which is continuous by question 4. Thus  $F$  is continuous and we are done.

- (c) This is a bit easier: let  $\Omega_{K,G} \subset \Omega Y$  be a sub-basic open set with  $K \subset I$  compact and  $G \subset Y$  open. Then  $\alpha \in \Omega(f)^{-1}(\Omega_{K,G})$  if and only if  $f \circ \alpha(K) \in G$  if and only if  $\alpha(K) \subset f^{-1}(G)$  if and only if  $\alpha \in \Omega_{K,f^{-1}(G)}$ . Thus  $\Omega(f)^{-1}(\Omega_{K,G}) = \Omega_{K,f^{-1}(G)}$  which is open. Thus  $\Omega(f)$  is continuous.
- (d) We therefore define  $f_* : \pi_1(\Omega X, e) \rightarrow \pi_1(\Omega Y, e)$  by  $f_* = \Omega(f)_*$  and the functorial properties are easy to check using  $\Omega(g \circ f) = \Omega(g) \circ \Omega(f)$ .