M40: Exercise sheet 7

1. Let $p : E \to X$ be a covering space and consider the action of $\pi_1(X, x_0)$ on $p^{-1}(\{x_0\})$. Show:
   (a) if E is path-connected, this action is *transitive*: that is, if $e_1, e_2 \in p^{-1}\{x_0\}$ then there is $[\sigma] \in \pi_1(X, x_0)$ with $e_1 \cdot [\sigma] = e_2$.
   (b) the stabiliser of $e \in p^{-1}(\{x_0\})$ is the subgroup $G^e = \{[\sigma] \in \pi_1(X, x_0) : e \cdot [\sigma] = e\}$. Show that
   $$G^e = p_* \pi_1(E, e) \subset \pi_1(X, x_0).$$
   (c) Let A be a set with a transitive right G-action with trivial stabilisers: $a \cdot g = a$, for some $a \in A$, if and only if $g = 1$. Then, for fixed $a \in A$, the map $G \to A : g \mapsto a \cdot g$ is a bijection.
   (d) Deduce that if E is simply connected then, for $e \in p^{-1}\{x_0\}$, the map $[\sigma] \to e \cdot [\sigma]$ is a *bijection* $\pi_1(X, x_0) \to p^{-1}\{x_0\}$.
2. Show that the deck translations of a covering space form a group under composition.
3. Consider the covering map $\phi : \mathbb{R} \to S^1$ given by $\phi(t) = e^{2\pi i t}$. What are the deck translations of ϕ?
4. Let E be simply connected and locally path-connected and let $p : (E, e_0) \to (X, x_0)$ be a covering map.
   Show that if $e_1, e_2 \in p^{-1}\{x_0\}$ then there is a *unique* deck translation $\phi : E \to E$ with $\phi(e_1) = e_2$.
   Hint: Use the Ultimate Lifting Theorem: a deck translation is a lift of p itself!
5. Let χ be the isomorphism between the group of deck translations of a simply connected covering space of X and $\pi_1(X, x_0)$. Show that χ is related to the action of $\pi_1(X, x_0)$ on $p^{-1}\{x_0\}$ by
   $$\phi(e_0) = e_0 \cdot (\chi(\phi)).$$
6. Let X be a path-connected topological space and $p : E \to X$ a covering map. Let $x, y \in X$.
   Show that there is a a bijection $p^{-1}\{x\} \cong p^{-1}\{y\}$. Thus all fibres of p have the same cardinality.
7. Let $p : (E, e_0) \to (X, x_0)$ be a covering map such that $p_* : \pi_1(E, e_0) \to \pi_1(X, x_0)$ is surjective and E is path-connected. Show that p is a homeomorphism.

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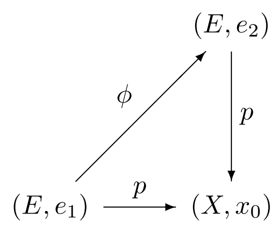
M40: Exercise sheet 7—Solutions

1. (a) Let $e_1, e_2 \in p^{-1}(\{x_0\})$ and let σ' be a path from e_1 to e_2. Then $\sigma = p \circ \sigma'$ is a loop at x_0 with $\sigma'_{e_1} = \sigma'$. Thus $e_1 \cdot [\sigma] = \sigma'(1) = e_2$ so that the action is transitive.
(b) $[\sigma]$ is in the stabiliser of e if and only if $\sigma'_e(1) = e$, that is, if and only if σ'_e is a loop at e or equivalently, $\sigma = p \circ \sigma'$ where σ' is a loop at e. Thus the stabiliser of e coincides with $p_*(\pi_1(E, e))$.
(c) If $b \in A$, by transitivity of the action, there is $g \in G$ with $b = a \cdot g$ so our map surjects. For injectivity, suppose that $a \cdot g_1 = a \cdot g_2$, then act on both sides by g_1^{-1}:
$$a = a \cdot 1 = (a \cdot g_1) \cdot g_1^{-1} = (a \cdot g_2) \cdot g_1^{-1} = a \cdot (g_2 g_1^{-1})$$
to get $a = a \cdot (g_2 g_1^{-1})$, that is, $g_2 g_1^{-1} \in G^a = \{1\}$. Thus $g_1 = g_2$.
(d) If E is simply connected, it is path-connected so that the action is transitive by (a). Moreover, from (b) we have that the action has trivial stabilisers so that (c) applies.

2. This is utterly trivial.

3. For $n \in \mathbb{Z}$, define the translation $T_n : \mathbb{R} \to \mathbb{R}$ by $T_n t = t + n$. Clearly $\phi \circ T_n(t) = e^{2\pi i(t+n)} = e^{2\pi i t} = \phi(t)$ so that each T_n is a deck translation. In fact these are all there are: if T is another deck translation, then $\phi(T(0)) = 1$ so that $T(0)$ is an integer n, say. But then T and T_n are both deck translations that agree at 0 and so agree everywhere since $\mathbb{R}$ is connected by the Unique Lifting Property (deck translations are lifts of ϕ).

4. We are seeking a homeomorphism $\phi : (E, e_1) \to (E, e_2)$ such that $p \circ \phi = p$. In particular, we are seeking a lift of p:



The ultimate lifting theorem guarantees that there is a (unique) continuous map ϕ with $p \circ \phi = p$ and all we need to do is check that ϕ is a homeomorphism. For this we use the ultimate lifting theorem to produce continuous $\psi : (E, e_2) \to (E, e_1)$ with $p \circ \psi = p$.
Now $\psi \circ \phi : (E, e_1) \to (E, e_1)$ is a lift of p but so is id_E (since id_E is a deck translation!). Thus, by unique lifting, $\psi \circ \phi = \mathrm{id}_E$. The same argument gives $\phi \circ \psi = \mathrm{id}_E$ so that ϕ is a homeomorphism (and so deck translation) with inverse ψ.

5. This is a matter of unravelling the definitions: $\chi(\phi) = [\sigma]$ where $\sigma = p \circ \sigma'$ for σ' a path from e_0 to $\phi(e_0)$. On the other hand, $e_0 \cdot [\sigma] = \sigma'_{e_0}(1)$. But, by uniqueness of lifts, $\sigma'_{e_0} = \sigma'$ so that $\sigma'_{e_0}(1) = \sigma'(1) = \phi(e_0)$.

6. Let σ be a path from x to y (which exists because X is path-connected) and define a map $\Phi_\sigma : p^{-1}\{x\} \to p^{-1}\{y\}$ by
$$\Phi_\sigma(e) = \sigma'_e(1) \in p^{-1}\{y\}.$$
To see that Φ_σ bijects, we show that it has inverse $\Phi_{\overline{\sigma}}$. Indeed, if $\Phi_\sigma(e) = e_1$, then $\sigma'_e \cdot \overline{\sigma}'_{e_1}$ is the lift starting at e of $\sigma \cdot \overline{\sigma}$.
However, unique lifting tells us that
$$\overline{\sigma}'_{e_1} = \overline{\sigma'_e}$$
since they both lift $\overline{\sigma}$ and agree at 0. In particular,
$$\overline{\sigma}'_{e_1}(1) = \overline{\sigma'_e}(1) = e$$
so that
$$e = \Phi_{\overline{\sigma}}(e_1) = \Phi_{\overline{\sigma}}(\Phi_\sigma(e))$$
and $\Phi_{\overline{\sigma}} \circ \Phi_\sigma = \mathrm{id}_{p^{-1}\{x\}}$. Now apply this with $\overline{\sigma}$ replacing σ to bake the cake!

7. Coverings maps are always open, continuous surjections so it suffices to prove that p is injective, that is $|p^{-1}\{x\}| = 1$, for all $x \in X$. In view of question 6 (which applies since E and so X is path-connected), it suffices to show that $|p^{-1}\{x_0\}| = 1$.
For this, $e_1 \in p^{-1}\{x_0\}$ and let σ' be a path from e_0 to e_1. Then $\sigma = p \circ \sigma'$ is a loop at x_0 and so, by hypothesis, there is a loop τ' at e_0 with $\sigma \sim p \circ \tau'$. But now homotopy lifting gives $\sigma' \sim \tau'$ and, in particular,

$$e_1 = \sigma'(1) = \tau'(1) = e_0.$$