## M40: Exercise sheet 6

1. Show that the retraction defined in the Brouwer Fixed Point Theorem is indeed continuous.

2. Let $X$ be homeomorphic to the closed unit disc in $\mathbb{R}^2$. Show that any continuous map $f : X \to X$ has a fixed point. (This is an *easy* application of Brouwer's theorem.)

3. Show that every $3 \times 3$ matrix with positive real entries has an eigenvector with positive eigenvalue.
   **Hint**: Contemplate the triangle $T = \{x + y + z = 1 \colon x,\, y,\, z \geq 0\}$ and the map of the positive quadrant onto $T$ given by $(x, y, z) \mapsto (x, y, z)/(x + y + z)$.

4. Let $p : E \to X$ be a covering map. Show:
   (a) For each $x \in X$, $p^{-1}\{x\}$ is discrete;
   (b) $p$ is open;
   (c) $p$ is a local homeomorphism.

5. Recall that $\mathbb{R}P^n$ is the topological quotient of $S^n$ by the relation $x \sim y$ if and only if $x = \pm y$.
   Let $p : S^n \to \mathbb{R}P^n$ be the projection map $p(x) = [x]$.
   (a) Show that $p$ is a covering map.
   (b) By contemplating the map $\psi : S^1 \to S^1 \subset \mathbb{C}$ given by $\psi(z) = z^2$, show that $\mathbb{R}P^1 \cong S^1$.

6. Let $X$ and $Y$ be topological spaces and $\mathcal{C}(X, Y)$ the set of continuous maps from $X$ to $Y$ equipped with the compact-open topology.
   (a) Let $f,\, g \in \mathcal{C}(X, Y)$ and suppose that $f \simeq g$. Show that there is a path in $\mathcal{C}(X, Y)$ joining $f$ and $g$.
   (b) Now assume that $X$ is *strongly locally compact*[1]. Show that, in this case, if there is a path joining $f$ and $g$ in $\mathcal{C}(X, Y)$ then $f \simeq g$.
   **Hint**: You may find a result from a previous sheet helpful here.

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[1]Recall that this means that every neighbourhood of $x \in X$ contains a compact neighbourhood of $x$

**M40: Exercise sheet 6—Solutions**

1. Let $D$ denote the closed unit ball in $\mathbb{R}^n$ and $f : D \to D$ a continuous map without fixed points. The retraction $r : D \to S^{n-1}$ is defined by setting $r(x)$ to be the intersection of the line from $f(x)$ through $x$ with the boundary of the ball. Thus $r(x) = (1-t)f(x)+tx = f(x)+t(x-f(x))$ where $t$ is positive and such that $|r(x)| = 1$. Squaring this last gives
$$1 = |x - f(x)|^2 t^2 + 2(f(x), x - f(x))t + |f(x)|^2$$
where $(\cdot, \cdot)$ is the inner product on $\mathbb{R}^n$. Thus the usual formula for solving a quadratic gives
$$t = \frac{-2(x, x - f(x)) \pm \sqrt{4(x, x - f(x))^2 - 4|x - f(x)|^2(|f(x)|^2 - 1)}}{2|x - f(x)|^2}.$$
Now $f(x) \in D$ so that $|f(x)|^2 - 1 \leq 0$ whence we see that
$$\sqrt{4(x, x - f(x))^2 - 4|x - f(x)|^2(|f(x)|^2 - 1)} \geq 2|(x, x - f(x))|$$
and thus that the positive solution is given by
$$t = \frac{-2(x, x - f(x)) + \sqrt{4(x, x - f(x))^2 - 4|x - f(x)|^2(|f(x)|^2 - 1)}}{2|x - f(x)|^2}$$
which is a (horrible, but) continuous function of $x$. Thus $r$ is continuous.

2. Let $\phi : D \to X$ be a homeomorphism and let $f : X \to X$ be continous. Then $\phi^{-1} \circ f \circ \phi$ is a map of $D$ to itself and so has a fixed point $y$, i.e, $\phi^{-1}(f(\phi(y))) = y$. From this we see that $\phi(y)$ is a fixed point of $f$.

3. Let $Q = \{(x, y, z) \in \mathbb{R}^3 \colon x,\, y,\, z \geq 0$ and not all $x, y, z$ are zero$\}$. Note that $T \subset Q$ and define $r : Q \to T$ by $r((x, y, z)) = (x, y, z)/(x + y + z)$. Now let $A$ be a matrix with positive entries: viewed as a linear map, $A(Q) \subset Q$ and so $r \circ A$ restricts to give a continuous map $T \to T$. Since $T$ is obviously homeomorphic to a disc, we may apply Brouwer's theorem to get a fixed point $\mathbf{v} = (x, y, z)$ with $r(A\mathbf{v}) = \mathbf{v}$. Thus $A\mathbf{v} = \lambda\mathbf{v}$ where $\lambda$ is the sum of the coefficients of $A\mathbf{v}$—a positive number.

4. (a) Let $\{S_\alpha\}$ be the sheets over some evenly covered neighbourhood of $x \in X$. Then if $e \in p^{-1}\{x\}$, $e$ lies in some sheet $S_\alpha$ and $S_\alpha \cap p^{-1}\{x\} = \{e\}$ otherwise $p$ would fail to be injective on $S_\alpha$. Thus $\{e\}$ is open in the induced topology on $p^{-1}\{x\}$ and so this set is discrete.
(b) Let $G$ be open in $E$ and $e \in G$ with $p(e) = x$. Then $x$ has an evenly covered open neighbourhood $U$ and $e$ lies in some sheet $S$ over $U$. Now $S \cap G$ is open and $p|S$ is a homeomorphism so that $p(S \cap G)$ is an open neighbourhood of $x$ contained in $p(G)$. Thus $p(G)$ is open.
(c) Any point in $e \in E$ has a neighbourhood which is a sheet over some evenly covered neighbourhood of $p(e)$ and $p$ is a homeomorphism on this sheet.

5. (a) First note that $p$ is open: if $U \subset S^n$ is open then $p^{-1}(p(U)) = U \cup -U$ where $-U = \{-u : u \in U\}$ and this is a union of open sets and so open whence $p(U)$ is open in the quotient topology.
Now let $[x] \in \mathbb{R}P^n$ and choose an open neighbourhood $S$ of $x \in S^n$ such that $S \cap -S = \emptyset$ (for example, take $S$ to be all $y \in S^n$ whose inner product with $x$ is strictly positive). Then $p$ is a continuous, open bijection from $S$ to $p(S)$ and similarly $p$ restricts to give a homeomorphism $-S \to p(S)$. Thus $p(S)$ is evenly covered with sheets $S$ and $-S$.
We conclude that $p$ is a covering map.
(b) Define $\phi : \mathbb{R}P^1 \to S^1$ by $\phi([x]) = x^2 \in S^1$. This is well-defined since $x^2 = (-x)^2$ and continuous since $\psi = \phi \circ p$. It is easy to check that $\phi$ is a bijection and, since $\mathbb{R}P^1$ is compact and $S^1$ is Hausdorff, it follows that $\phi$ is a homeomorphism.

6. (a) Let $f \simeq g$ via a homotopy $F$ and for $t \in I$ define $F_t : X \to Y$ by $F_t(x) = F(x,t)$. Clearly each $F_t \in \mathcal{C}(X,Y)$ and we define $\gamma : I \to \mathcal{C}(X,Y)$ by $\gamma(t) = F_t$. Clearly, $\gamma(0) = f$ and $\gamma(1) = g$ so all we need to do is show that $\gamma$ is continuous to see that it is the required path. For this, it suffices to show that the inverse image by $\gamma$ of any sub-basic open set of $\mathcal{C}(X,Y)$ is open. So let $\mathcal{C}_{K,G}$ be such a set and consider its inverse image $V = \gamma^{-1}(\mathcal{C}_{K,G})$, thus $t \in V$ if and only if $F(x,t) \in G$ for all $x \in K$. So fix $t \in V$ and let $x \in K$. By the continuity of $F : X \times I \to Y$, each $(x,t)$ is contained in a basic open set $U_x \times I_x$ of $X \times I$ which is contained in $F^{-1}(G)$. Now $\{U_x\}_{x \in K}$ is an open cover of $K$ so we have a finite sub-cover $U_{x_1}, \ldots, U_{x_n}$, say. Let $J = \bigcap_{i=1}^n I_{x_i}$—an open neighbourhood of $t$. I claim that $J \subset V$ from which it follows that $V$ is open and $\gamma$ is continuous. For the claim, let $s \in J$ and $x \in K$, we must show that $F(x,s) \in G$. But $x \in U_{x_i}$ for some $i$ so that $(x,s) \in J \times U_{x_i} \subset I_{x_i} \times U_{x_i} \subset F^{-1}(G)$ and the claim is proved.

(b) For the converse, let $\gamma$ be a path in $\mathcal{C}(X,Y)$ from $f$ to $g$. The obvious thing to do is to define $F : X \times I \to Y$ by $F(x,t) = \gamma(t)(x)$: clearly $F(x,0) = f(x)$ and $F(x,1) = g(x)$ so, just like last time, the thing to do is show continuity of $F$.

Recall from a previous sheet that the map $\mathrm{ev} : X \times \mathcal{C}(X,Y) \to Y$ given by $\mathrm{ev}(x,\phi) = \phi(x)$ is continuous for $X$ strongly locally compact. Further our proposed homotopy $F$ is the composition of ev with the continuous map $(x,t) \mapsto (x,\gamma(t)) : X \times I \to X \times \mathcal{C}(X,Y)$ and so is continuous.