M40: Exercise sheet 5

1. Let X be a topological space. Recall that the *path components* of X are the equivalence classes in X under the equivalence relation where x and y are related if and only if there is a path from x to y. (For X locally path-connected, these are just the connected components of X.) Denote the set of path components of X by $\pi_0(X)$. Now let $\phi : X \to Y$ be a continuous map.
   (a) Define a map $\phi_* : \pi_0(X) \to \pi_0(Y)$. (In particular, show it is well-defined).
   (b) Show that if $\phi \simeq \psi$ then $\phi_* = \psi_*$.
   (c) Show that if $f : X \to Y$ and $g : Y \to Z$ are continuous then $(g \circ f)_* = g_* \circ f_*$ while $(\mathrm{id}_X)_* = \mathrm{id}_{\pi_0(X)}$.

   (Thus we get another example of a *functor*: this time from topological spaces to sets.)

   (d) Finally prove that if ϕ is a homotopy equivalence then ϕ_* is a bijection.

2. Let X have the property that every continuous map $S^1 \to X$ is homotopic to a constant. Prove that for any $x \in X$, $\pi_1(X, x) = \{1\}$. **Hint**: View a loop as a map $S^1 \to X$ (but be careful about basepoints!).

3. (Lebesgue Covering Lemma) Let (X, d) be a compact metric space and $\{U_\alpha\}$ an open cover of X. Show that there is a number $\delta > 0$ such that any subset of X of diameter less than δ is completely contained in some U_α.

4. Let X, Y be topological spaces with Y Hausdorff. Show that $\mathcal{C}(X, Y)$ (with the compact-open topology) is Hausdorff also.

5. Let X, Y be topological spaces and, as usual, give $\mathcal{C}(X, Y)$ the compact-open topology.
   (a) (Easy) For $x \in X$, define $\mathrm{ev}_x : \mathcal{C}(X, Y) \to Y$ by

   $$\mathrm{ev}_x(\phi) = \phi(x).$$

   Show that ev_x is continuous.

   (b) (A little harder) Now suppose that X has the property that any neighbourhood of any $x \in X$ contains a compact neighbourhood of x (we say that X is *strongly locally compact*[1]). Define $\mathrm{ev} : X \times \mathcal{C}(X, Y) \to Y$ by

   $$\mathrm{ev}(x, \phi) = \phi(x).$$

   Show that ev is continuous (here, of course, we give $X \times \mathcal{C}(X, Y)$ the product topology).

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[1] Observe that any interval in $\mathbb{R}$ has this property.

M40: Exercise sheet 5—Solutions

1. (*a*) Let C_x denote the path component of x and define $\phi_* : \pi_0(X) \to \pi_0(Y)$ by $\phi_* C_x = C_{\phi(x)}$. To see that this is well defined, suppose $C_{x_1} = C_{x_2}$. This means there is a path, γ say, from x_1 to x_2. But then $\phi \circ \gamma$ is a path from $\phi(x_1)$ to $\phi(x_2)$ so that $C_{\phi(x_1)} = C_{\phi(x_2)}$.

   (*b*) Let F be a homotopy from ϕ to ψ and let $x \in X$. Then $t \mapsto F(x,t)$ is a path from $\phi(x)$ to $\psi(x)$ so that $C_{\phi(x)} = C_{\psi(x)}$, i.e., $\phi_*(C_x) = \psi_*(C_x)$ as required.

   (*c*) Fix $x \in X$. Then $g_* \circ f_*(C_x) = g_*(C_{f(x)}) = C_{g(f(x))} = (g \circ f)_*(C_x)$. Moreover, $(\mathrm{id}_X)_*(C_x) = C_x = \mathrm{id}_{\pi_0(X)}(C_x)$.

   (*d*) Finally, if $f : X \to Y$ is a homotopy equivalence with homotopy inverse g then $g \circ f \simeq \mathrm{id}_X$ so that $g_* \circ f_* = (g \circ f)_* = (\mathrm{id}_X)_* = \mathrm{id}_{\pi_0(X)}$ and, similarly, $f_* \circ g_* = \mathrm{id}_{\pi_0(Y)}$. Thus f_* is a bijection with inverse g_*.

2. We may identify loops in X with maps of S^1 into X as follows: if α is a loop in X, define $\hat{\alpha} : S^1 \to X$ by $\hat{\alpha}(e^{2\pi it}) = \alpha(t)$. Then $\hat{\alpha}$ is continuous by the universal property of quotients: recall that S^1 has the quotient topology induced by $t \mapsto e^{2\pi it} : I \to S^1$.
   With this in hand, suppose now that α is a loop at x and that $\hat{\alpha}$ is homotopic to a constant map with value y, say, via a homotopy $\hat{F} : S^1 \times I \to X$. Then $\alpha \simeq \gamma_y$ via F given by $F(t,s) = \hat{F}(e^{2\pi it}, s)$. Note that $F(0,s) = F(1,s)$ for each $s \in I$. Let β be the path from x to y given by $\beta(s) = F(0,s)$ then, from lemma (2.13) of the lecture notes, we have
   $$\alpha \sim \beta \cdot \gamma_y \cdot \overline{\beta} \sim \gamma_x,$$
   as required.

3. Let $x \in X$. Then $x \in U_\alpha$ for some $\alpha(x)$ and there is $\delta_x > 0$ so that the $2\delta_x$ ball $B_{2\delta_x}(x)$ about x is contained in U_α. Then $\{B_{\delta_x}(x)\}_{x \in X}$ is an open cover for X and so there is a finite sub-cover $\{B_{\delta_{x_i}}(x_i)\}_{1 \le i \le n}$.
   Now set $\delta = \min_{1 \le i \le n} \delta_i > 0$ and suppose that $A \subset X$ has diameter less than δ. Fix an $a \in A$. Then $a \in B_{\delta_{x_i}}(x_i)$ for some i whence the triangle inequality gives $A \subset B_{2\delta_{x_i}}(x_i)$ which last lies completely in some U_α as required.

4. Suppose that $f \neq g \in \mathcal{C}(X,Y)$. Thus, for some $x \in X$, $f(x) \neq g(x)$. Since Y is Hausdorff, there are disjoint open sets G_1 and G_2 in Y with $f(x) \in G_1$ and $g(x) \in G_2$.
   Contemplate now the subbasic open sets $\mathcal{C}_{\{x\},G_i}$, $i = 1,2$, of $\mathcal{C}(X,Y)$. We have $f \in \mathcal{C}_{\{x\},G_1}$, $g \in \mathcal{C}_{\{x\},G_2}$ and
   $$\mathcal{C}_{\{x\},G_1} \cap \mathcal{C}_{\{x\},G_2} = \mathcal{C}_{\{x\},G_1 \cap G_2} = \emptyset.$$
   Thus $\mathcal{C}(X,Y)$ is Hausdorff.

5. (i) Let G be open in Y. Then
   $$\mathrm{ev}_x^{-1}(G) = \{f \colon f(x) \in G\} = \mathcal{C}_{\{x\},G}$$
   is a subbasic open set so that ev_x is indeed continuous.
   (ii) Again let Y be open in G and contemplate
   $$\mathrm{ev}^{-1}(G) = \{(x,f) \colon f(x) \in G\}.$$
   Fix $(x,f) \in \mathrm{ev}^{-1}(G)$. Then, since f is continuous, $f^{-1}(G)$ is an open neighbourhood of x and so contains a compact neighbourhood K of x since X is strongly locally compact. So we have an open subset U of X with $x \in U \subset K$ and we can form the basic open set $U \times \mathcal{C}_{K,G}$ of $X \times \mathcal{C}(X,Y)$.
   By construction, $(x,f) \in U \times \mathcal{C}_{K,G}$ while, for any $(y,g) \in U \times \mathcal{C}_{K,G}$, we have $y \in K$ so that $g(y) \in G$ so that
   $$(x,f) \in U \times \mathcal{C}_{K,G} \subset \mathrm{ev}^{-1}(G).$$
   It follows at once that $\mathrm{ev}^{-1}(G)$ is open in $X \times \mathcal{C}(X,Y)$.