M40: Exercise sheet 4

1. Let ϕ_1, $\phi_2 : X \to Y$ and ψ_1, $\psi_2 : Y \to Z$ be continuous maps. If $\phi_1 \simeq \phi_2$ and $\psi_1 \simeq \psi_2$, show that $\psi_1 \circ \phi_1 \simeq \psi_2 \circ \phi_2$. Deduce that homotopy equivalence is an equivalence relation on topological spaces.

2. (*a*) Let $A \subset \mathbb{R}^n$ be convex and $a \in A$. Show that $\{a\}$ is a deformation retract of A.

   (*b*) Show that S^n is a deformation retract of $\mathbb{R}^{n+1} \setminus \{0\}$.

3. Let X be a topological space.

   (*a*) Show that X is contractible if and only if the identity map id_X is homotopic to a constant map.

   (*b*) Show that if X is contractible then X is simply connected.

   (*c*) Show that X is contractible if and only if any continuous map with X as domain or codomain is homotopic to a constant.

   (*d*) Let Y be contractible. Show that $X \simeq X \times Y$.

4. Recall that the discrete topology on a set A is the topology where all subsets are open. A subset A of a topological space X is said to be *discrete* if the induced topology on A is the discrete topology on A.

   (*a*) Show that $A \subset X$ is discrete if and only if each $a \in A$ has an open neighbourhood U in X such that $A \cap U = \{a\}$.

   (*b*) Deduce that $\mathbb{Z}$ is a discrete subset of $\mathbb{R}$.

   (*c*) Show that a continuous map of a connected space with discrete image is constant.

5. Give your favourite alphabet the induced topology from $\mathbb{R}^2$. Now divide it into classes of homotopy equivalent letters.

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Home page: `http://go.bath.ac.uk/ma40040`

M40: Exercise sheet 4—Solutions

1. Let F be a homotopy from ϕ_1 to ϕ_2 and G a homotopy from ψ_1 to ψ_2. Then setting $H(x,t) = G(F(x,t),t)$ gives a homotopy from $\psi_1 \circ \phi_1$ to $\psi_2 \circ \phi_2$.
   Now we can show that homotopy equivalence of topological spaces is an equivalence relation. First, for any X, id_X is a homeomorphism and so a homotopy equivalence so the relation is reflexive. Secondly, if $\phi : X \to Y$ is a homotopy equivalence with homotopy inverse $\psi : Y \to X$ then ψ is a homotopy equivalence with homotopy inverse ϕ so that the relation is symmetric. Lastly, it is easy to see from the above that a composition of homotopy equivalences is also a homotopy equivalence so that the relation is transitive.

2. (*a*) Let $i : \{a\} \to A$ be the inclusion and define $r : A \to \{a\}$ the only way you can: $r(x) = a$ for all $x \in A$. Now define $F : A \times I \to A$ by $F(x,t) = (1-t)x + ta$ (well-defined since A is convex) to get a homotopy (rel $\{a\}$) from id_A to $i \circ r$.

   (*b*) Let $r : \mathbb{R}^{n+1} \setminus \{0\} \to S^n$ by $r(x) = x/|x|$—this is clearly a retraction. Define $F : \mathbb{R}^{n+1} \setminus \{0\} \times I \to \mathbb{R}^{n+1} \setminus \{0\}$ by
   $$F(x,t) = (1-t)x + t\frac{x}{|x|}$$
   to get a homotopy (rel S^n) between the identity of $\mathbb{R}^{n+1} \setminus \{0\}$ and $i \circ r$.

3. (*a*) Let $\phi : X \to \{\mathrm{pt}\}$ be a homotopy equivalence with homotopy inverse ψ. Then $\psi \circ \phi$ is a constant map which, by definition, is homotopic to id_X.
   Conversely, if id_X is homotopic to a constant map $\phi : X \to X$ with value $x_0 \in X$, say, let $\psi : \{x_0\} \to X$ be the inclusion and observe that ϕ is a homotopy equvalence with homotopy inverse ψ.

   (*b*) A contractible space X is, by definition, homotopy equivalent to a singleton space $\{\mathrm{pt}\}$, say. In particular, since $\{x_0\}$ has trivial π_1 (the only loop is the constant loop), we conclude that X has trivial π_1. Moreover, if F is a homotopy between id_X and the constant map on X with value x_0, then, for any $x \in X$, $t \mapsto F(x,t)$ is a path from x to x_0 so that X is path-connected. Thus X is simply connected.

   (*c*) We know that if X is contractible then id_X is homotopic to a constant map c, say. Thus if $f : X \to Y$ then $f = f \circ \mathrm{id}_X \simeq f \circ c$ which last is a constant. A similar argument establishes that any $g : Y \to X$ is homotopic to a constant.

   (*d*) We know that there is a constant map $r : Y \to Y$ with value y_0, say, which is homotopic to id_Y via a homotopy F say. Now let $f : X \to X \times Y$ be given by $f(x) = (x, y_0)$ and $\pi : X \times Y \to X$ be projection onto the first factor: I claim that f is a homotopy equivalence with homotopy inverse π. Indeed, $\pi \circ f = \mathrm{id}_X$ while $G((x,y),t) = (x, F(y,t))$ is easily checked to be a homotopy from $f \circ \pi$ to $\mathrm{id}_{X \times Y}$.

4. (*a*) $A \subset X$ has discrete induced topology if and only if, for each $a \in A$, $\{a\}$ is open in A. But this is the case if and only if $\{a\} = U \cap A$ for some open set U in X.

   (*b*) Let $n \in \mathbb{Z}$. Then $\{n\} = \mathbb{Z} \cap (n-1, n+1)$ so that $\mathbb{Z}$ is discrete in $\mathbb{R}$.

   (*c*) Let $\phi : X \to Y$ be a continuous map with discrete image where X is connected. Let y be in the image of ϕ and suppose that ϕ is not constant. Then $\phi^{-1}(\{y\})$ and $\phi^{-1}(Y \setminus \{y\})$ are non-empty, disjoint and open (indeed, the inverse image of *any* set is open) and so disconnect X. Thus we conclude that ϕ is constant.

5. We take as our alphabet:

   A B C D E F G H I J K L M N O P Q R S T U V W X Y Z

   and then the separate homotopy equivalence classes are

   {A D O P Q R} {B} {C E F G H I J K L M N S T U V W X Y Z}

Note that the first set all have the homotopy type of the circle, the second that of a figure eight while the last set are all contractible.