M40: Exercise sheet 2

Fun with topological groups

1. Let X, Y be topological spaces and let $A \subset X$, $B \subset Y$ have the induced topology.
   (a) Show that the product topology on $A \times B$ is the same as the topology on $A \times B$ induced by the product topology on $X \times Y$.
   (b) If $f : X \to Y$ is continuous and $f(A) \subset B$, show that $f|A : A \to B$ is continuous with respect to the induced topologies on A and B.

   Deduce that a subgroup of a topological group becomes a topological group when equipped with the induced topology.

2. (Some subgroups of $\mathrm{GL}(n, \mathbb{R})$) Let $\mathrm{SL}(n)$ denote the set of $n \times n$ matrices with determinant 1, $\mathrm{O}(n)$ those which satisfy $AA^T = \mathrm{id}$ and put $\mathrm{SO}(n) = \mathrm{SL}(n) \cap \mathrm{O}(n)$. Show that these are all closed subgroups of $\mathrm{GL}(n, \mathbb{R})$. (They are called, respectively, the *special linear* group, the *orthogonal* group and the *special orthogonal* group.)

3. Prove that $\mathrm{SO}(n)$ and $\mathrm{O}(n)$ are compact.

4. Is $\mathrm{GL}(n, \mathbb{R})$ connected?

On paths and homotopy

5. Show that a path-connected space is connected.

6. (a) Define a relation on a topological space X by saying $x \sim y$ if and only if there is a path from x to y. Show that this is an equivalence relation.
   The equivalence classes are called the *path components* of X. Clearly X is path-connected if and only if X has exactly one path component.
   (b) Show that if X is locally path-connected then each path component is open.
   (c) Deduce that a connected and locally path-connected space is path-connected.
   (d) Show that an open set in $\mathbb{R}^n$ is locally path-connected (and so is connected if and only if path-connected).

7. Let $\alpha : [0, 1] \to X$ be a path and define $\overline{\alpha} : [0, 1] \to X$ by $\overline{\alpha}(t) = \alpha(1 - t)$. If $\alpha \sim \beta$, show that $\overline{\alpha} \sim \overline{\beta}$.

8. Let $\beta \in \mathrm{Path}(X, y, x)$ and let $\gamma_x \colon t \mapsto x$ be the constant loop at x. Write down a homotopy between β and $\beta \cdot \gamma_x$ (and prove it is one!).

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M40: Exercise sheet 2—Solutions

1. (a) I give two arguments: first let us show with our bare-hands that the topologies coincide: Let $E \times F$ be a basic open set in the product topology on $A \times B$ (thus E is open in A and F is open in B). So $E = A \cap U$, for some U open in X, and $F = B \cap V$ with V open in Y. Thus

$$E \times F = (A \cap U) \times (B \cap V) = (A \times B) \cap (U \times V)$$

so that, since $U \times V$ is open in $X \times Y$, we see that $E \times F$ is open in the induced topology on $A \times B$. Since any open set in the product topology on $A \times B$ is a union of such basic open sets, we conclude that all open sets in the product topology are open in the induced topology.

For the converse, let G be open in the induced topology. Thus $G = (A \times B) \cap \Omega$ where Ω is open in $X \times Y$. Then Ω is a union of basic open sets: $\Omega = \bigcup_\alpha U_\alpha \times V_\alpha$ with U_α open in X and V_α open in Y so that

$$G = (A \times B) \cap \bigcup_\alpha U_\alpha \times V_\alpha = \bigcup_\alpha (A \times B) \cap (U_\alpha \times V_\alpha) = \bigcup_\alpha (A \cap U_\alpha) \times (B \cap V_\alpha),$$

which last is a union of sets open in the product topology on $A \times B$. Thus G is open in that topology.

Alternatively, equip $A \times B$ with the induced topology from $X \times Y$. We show that this topology has the universal property of the product topology on $A \times B$. So let $f = (f_1, f_2) : Z \to A \times B$ be a map. Then f is continuous if and only if it is continuous as a map into $X \times Y$ (universal property of the induced topology on $A \times B$) if and only if f_1, f_2 are continuous as maps into X, Y, respectively, (universal property of the product topology on $X \times Y$) if and only if f_1, f_2 are continuous as maps into A, B (universal property of the induced topologies on A, B). Thus $f = (f_1, f_2) : Z \to A \times B$ is continuous if and only if $f_1 : Z \to A$ and $f_2 : Z \to B$ are continuous so that our topology on $A \times B$ is the product topology.

(b) Let G be open in B so that $G = B \cap U$ for U open in Y. Since $f(A) \subset B$, $(f|A)^{-1}(G) = (f|A)^{-1}(U) = A \cap f^{-1}(U)$ which last is open in A since f is continuous.

Now let H be a subgroup of a topological group and give H the induced topology. Inversion on H is the restriction of inversion on G and so is continuous by (b). Moreover, by (a), the product topology on $H \times H$ coincides with the topology induced by the product topology on $G \times G$ while the multiplication map on $H \times H$ is the restriction of that on $G \times G$ and so, again by (b), is continuous. Finally, the topology of H is Hausdorff since that of G is.

2. It is a straightforward algebraic matter to see that these subsets are indeed subgroups so I shall concentrate on the topological question of closure of these subgroups.

Firstly, $\mathrm{SL}(n) = \det^{-1}\{1\}$ is the inverse image of a closed set by the continuous map det and so is closed.

For $\mathrm{O}(n)$, contemplate the map $t : M(n) \to M(n)$ given by $A \mapsto AA^T$: each entry of AA^T is polynomial in the entries of A and so t is continuous. Moreover, $\mathrm{O}(n) = t^{-1}\{\mathrm{id}\}$ and so is closed as before.

Finally $\mathrm{SO}(n)$ is the intersection of two closed sets and so is closed.

3. In view of question 2 and the Heine–Borel theorem, it suffices to show that $\mathrm{O}(n)$ (and hence $\mathrm{SO}(n)$) is a bounded subset of $M(n) = \mathbb{R}^{n^2}$.

For this, it suffices to note that the condition $AA^T = \mathrm{id}$ amounts to the assertion that the rows of A constitute an orthonormal basis of $\mathbb{R}^n$. In particular, the sum of the squares of the entries in each row is 1 whence, summing over the n rows

$$\sum_{i,j} a_{ij}^2 = n.$$

Otherwise said, each $A \in \mathrm{O}(n)$ lies in the sphere of radius $\sqrt{n}$ about the origin in $\mathbb{R}^{n^2}$ whence $\mathrm{O}(n)$ is bounded and so compact.

4. No: $\mathrm{GL}(n, \mathbb{R})$ is the disjoint union of the non-empty open sets:

$$\mathrm{GL}^+(n) = \{A \in \mathrm{GL}(n, \mathbb{R}) \colon \det A > 0\} \qquad \mathrm{GL}^-(n) = \{A \in \mathrm{GL}(n, \mathbb{R}) \colon \det A < 0\}$$

(these are open being the inverse image by det of open intervals in $\mathbb{R}$ and non-empty since the identity matrix is in $\mathrm{GL}^+(n)$ while the matrix obtained by replacing the first entry of the identity matrix by -1 is in $\mathrm{GL}^-(n)$).

5. Let X be path-connected and suppose, for a contradiction, that X is not connected. So there are disjoint open sets G_1, G_2 such that $X = G_1 \cup G_2$. Choose $x \in G_1$, $y \in G_2$ and let $\gamma : I \to X$ be a path from x to y. Then $\gamma^{-1}(G_1)$ and $\gamma^{-1}(G_2)$ are non-empty disjoint open subsets of I whose union is I so that I is not connected. This is the required contradiction.

6. (a) We check the conditions: the constant path γ_x joins x to x so $x \sim x$ for any $x \in X$; if α is a path from x to y then $\overline{\alpha}$ is a path from y to x so that $x \sim y$ implies $y \sim x$; if α is a path from x to y and β is a path from y to z then $\alpha \cdot \beta$ is a path from x to z so that $x \sim y$ and $y \sim z$ imply $x \sim z$.
(b) Now suppose X is locally path connected and consider the path component C_x containing x. Let $y \in C_x$. Then there is a path-connected neighbourhood N_y of y so that, for all $z \in N_y$, $z \sim y$. Transitivity of $\sim$ now gives $z \sim x$ so that $N_y \subset C_x$. Thus each point of C_x has a neighbourhood also in C_x which forces C_x to be open.
(c) If we additionally demand that X be connected, we see that there can only be one path component since otherwise, we would have that X is a disjoint union of non-empty open sets (viz. the path components). Thus X is path-connected.
(d) Note that any ε-ball in $\mathbb{R}^n$ is path-connected (if $x, y \in B_\varepsilon(z)$ then the path $t \mapsto (1-t)x + ty$ lies in $B_\varepsilon(z)$) so that any open set in $\mathbb{R}^n$ is locally path-connected and so connected if and only if path-connected.

7. Let $F : I \times I \to X$ be the homotopy between α and β and define $\widetilde{F} : I \times I \to X$ by $\widetilde{F}(t, s) = F(1-t, s)$. That this is a homotopy between $\overline{\alpha}$ and $\overline{\beta}$ is the work of seconds to prove.

8. I'll just write down the answer: define $F : I \times I \to X$ by

$$F(t, s) = \begin{cases} \beta(2t/(2-s)) & \text{for } 0 \le t \le 1 - s/2; \\ x & \text{for } 1 - s/2 \le t \le 1. \end{cases}$$

The usual arguments demonstrate that this is the required homotopy.