M40: Exercise sheet 1 (mostly revision)

1. Let X be a topological space and A, B be closed subsets of X such that $A \cup B = X$. Let $\phi : X \to Y$ be a map into a topological space Y such that $\phi|A$ and $\phi|B$ are continuous with respect to the induced topologies on A and B respectively. Show that ϕ is continuous on X.

2. We establish some universal properties:

   (a) Let X be a topological space and $\mathcal{T}_A$ the induced topology on $A \subset X$. Let $i : A \hookrightarrow X$ be the inclusion.
   Show that $\mathcal{T}_A$ is the *unique* topology on A such that, for any topological space Y and map $f : Y \to A$, f is continuous if and only if $i \circ f$ is continuous.

   (b) Let $X_1, \ldots, X_n$ be topological spaces.
   Show that the product topology is the unique topology on $X_1 \times \cdots \times X_n$ with the property that, for any topological space Y and map $f : Y \to X_1 \times \cdots \times X_n$, f is continuous if and only if each component $\pi_i \circ f : Y \to X_i$ is continuous.

   (c) Let X be a topological space and $\pi : X \to Y$ a surjection onto a set Y.
   Show that the quotient topology $\mathcal{T}_\pi$ is the *unique* topology on Y with the property that, for all topological spaces Z and maps $f : Y \to Z$, f is continuous if and only if $f \circ \pi$ is continuous.

3. Let $f : \mathbb{R}^2 \to \mathbb{R}$ have the property that, for each x, $y \in \mathbb{R}$, the functions f_x, $f_y : \mathbb{R} \to \mathbb{R}$ given by $f_x(z) = f(x, z)$, $f_y(z) = f(z, y)$ are continuous. Is f continuous? Give a proof or find a counter-example.

4. Let X be compact, Y Hausdorff and $\phi : X \to Y$ a continuous bijection. Show that ϕ is a homeomorphism.

5. Let X and Y be topological spaces and $f : X \to Y$, $g : Y \to X$ continuous maps such that $f \circ g = \mathrm{id}_Y$. Prove

   (a) f is surjective and g is injective;

   (b) the topology of Y is the quotient topology induced by f;

   (c) g is a homeomorphism from Y to $g(Y)$ (with topology induced from X);

   (d) if X is Hausdorff, so is Y.

6. Let X, Y be topological spaces and $\pi : X \times Y \to Y$ be the natural projection. Show that the topology of Y is the quotient topology induced by π.

7. Give your favorite alphabet the induced topology from $\mathbb{R}^2$. Now divide it into classes of homeomorphic letters.

October 8, 2018

M40: Exercise sheet 1 (mostly revision)—Solutions

1. We show that the inverse image of a closed set is closed. So let $F \subset Y$ be closed and consider $\phi^{-1}(F) = (\phi|A)^{-1}(F) \cup (\phi|B)^{-1}(F)$. Now $(\phi|A)^{-1}(F)$ is closed in the induced topology on A and so is of the form $A \cap F'$ for some closed subset F' of X. But A is itself closed in X whence $A \cap F'$ is closed in X, that is $(\phi|A)^{-1}(F)$ is closed in X. Similarly, $(\phi|B)^{-1}(F)$ is closed in X so that their union $\phi^{-1}(F)$ is closed and ϕ is continuous as required.

2. (*a*) We prove uniqueness first: let $\mathcal{T}_1, \mathcal{T}_2$ be topologies on A with the universal property. Now $\mathrm{id} : (A, \mathcal{T}_1) \to (A, \mathcal{T}_1)$ is continuous so that (universal property of $\mathcal{T}_1$), $i = i \circ \mathrm{id} : (A, \mathcal{T}_1) \to X$ is continuous. Now the universal property of $\mathcal{T}_2$ says that $\mathrm{id} : (A, \mathcal{T}_1) \to (A, \mathcal{T}_2)$ is continuous, that is, $\mathcal{T}_2 \subset \mathcal{T}_1$. Now swop the roles of $\mathcal{T}_1$ and $\mathcal{T}_2$ to get equality.
   Now we establish that $\mathcal{T}_A$ has the universal property. Begin by observing that i is continuous: for $G \in \mathcal{T}$, $i^{-1}G = A \cap G \in \mathcal{T}_A$. Now let $f : Y \to A$ be a map. If f is continuous then so is $i \circ f$, being a composition of continuous maps. Conversely, if $i \circ f$ is continuous and $B \in \mathcal{T}_A$, then $B = A \cap G$, for some $G \in \mathcal{T}$, so that
   $$f^{-1}B = f^{-1}(i^{-1}G) = (i \circ f)^{-1}G$$
   is open in Y and f is continuous.

   (*b*) Uniqueness first: write X for the product $X_1 \times \cdots \times X_n$ and let $\mathcal{T}_1, \mathcal{T}_2$ be topologies on X with the universal property. Now $\mathrm{id} : (X, \mathcal{T}_1) \to (X, \mathcal{T}_1)$ is continuous so that (universal property of $\mathcal{T}_1$), each $\pi_i = \pi_i \circ \mathrm{id} : (X, \mathcal{T}_1) \to X_i$ is continuous. Now the universal property of $\mathcal{T}_2$ says that $\mathrm{id} : (X, \mathcal{T}_1) \to (X, \mathcal{T}_2)$ is continuous, that is, $\mathcal{T}_2 \subset \mathcal{T}_1$. Now swop the roles of $\mathcal{T}_1$ and $\mathcal{T}_2$ to get equality. (Note how this argument is essentially a cut-and-paste of the uniqueness argument for the induced topology.)
   Now let $\mathcal{T}$ be the product topology. Then each π_i is continuous with respect to $\mathcal{T}$ since, for G_i open in X_i,
   $$\pi^{-1}G_i = X_1 \times \cdots \times G_i \times \cdots \times X_n \in \mathcal{T}.$$
   If $f : Y \to X$ is continuous (with respect to $\mathcal{T}$), so is each $\pi_i \circ f$. Conversely, if $f_i = \pi \circ f$ is continuous and $G = G_1 \times \cdots \times G_n$ is a basic open subset of X, then
   $$f^{-1}G = \bigcap_{i=1}^{n} (\pi \circ f)^{-1} G_i,$$
   which is open in Y so that f is continuous.

   (*c*) Again, uniqueness first. Let $\mathcal{T}_1, \mathcal{T}_2$ be topologies on Y with the universal property. Now $\mathrm{id} : (Y, \mathcal{T}_1) \to (Y, \mathcal{T}_1)$ is continuous so that (universal property of $\mathcal{T}_1$), $\pi = \mathrm{id} \circ \pi : X \to (Y, \mathcal{T}_1)$ is continuous. Now the universal property of $\mathcal{T}_2$ says that $\mathrm{id} : (Y, \mathcal{T}_2) \to (Y, \mathcal{T}_1)$ is continuous, that is, $\mathcal{T}_1 \subset \mathcal{T}_2$. Now swop the roles of $\mathcal{T}_1$ and $\mathcal{T}_2$ to get equality.
   To see that $\mathcal{T}_\pi$ has the universal property, first note that, by definition, π is continuous $(X, \mathcal{T}) \to (Y, \mathcal{T}_\pi)$. So it is certainly true that if $f : Y \to Z$ is continuous then so is $f \circ \pi$. For the converse, if $f \circ \pi : X \to Z$ is continuous and $G \subset Z$ is open, then $\pi^{-1}(f^{-1}G) = (f \circ \pi)^{-1}G \in \mathcal{T}$, that is, $f^{-1}G \in \mathcal{T}_\pi$.

3. Here is a counter-example: define $f : \mathbb{R}^2 \to \mathbb{R}$ by
   $$f(x, y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{for } (x, y) \neq (0, 0); \\ 0 & \text{for } (x, y) = (0, 0). \end{cases}$$
   It is easy to see that each f_x and f_y are continuous (indeed, $f_x(y) = xy/(x^2 + y^2)$ for all y when $x \neq 0$ and $f_x \equiv 0$ when $x = 0$, for example.). However, $f(x, x) = \frac{1}{2}$ for $x \neq 0$ and 0 when $x = 0$ so f is not continuous on $\mathbb{R}^2$.

4. All we need show is that ϕ^{-1} is continuous or, equivalently, that ϕ sends closed sets in X to closed sets in Y. But closed subsets of X are compact since X is compact so that their images by the continuous map ϕ are compact and hence closed since Y is Hausdorff.

5. (a) It is always the case that if $f \circ g$ is a bijection then f is surjective and g is injective.

   (b) We show that $f : X \to Y$ has the universal property of quotients: so let $h : Y \to Z$ be a map into a space Z. If h is continuous then clearly $h \circ f$ is continuous (composition of continuous maps is continuous) while, if $h \circ f$ is continuous then so is $h = (h \circ f) \circ g$ for the same reason.

   If that is all a bit too slick, here is a down-to-earth argument. Let $\mathcal{S}$ be the given topology on Y and $\mathcal{T}_f$ the quotient topology induced by f. Now, if $G \in \mathcal{S}$ then $f^{-1}(G)$ is open in X (by continuity of f) so that $G \in \mathcal{T}_f$.

   Conversely, if $G \in \mathcal{T}_f$ then, by definition of the quotient topology, $f^{-1}(G)$ is open in X so that, by continuity of g, $g^{-1}(f^{-1}(G)) \in \mathcal{S}$. But $g^{-1}(f^{-1}(G)) = (f \circ g)^{-1}(G) = G$ so we are done.

   (c) g is certainly a continuous bijection onto its image and moreover its inverse is just $f|g(Y)$ which is continuous in the induced topology (as all restrictions of maps continuous on X are). Thus g is a homeomorphism onto its image.

   (d) If X is Hausdorff, so is any subset with the induced topology. Thus $g(Y)$ is Hausdorff and we have just shown that this is homeomorphic to Y whence Y is Hausdorff also.

6. Fix $x \in X$ and define $g : Y \to X \times Y$ by $g(y) = (x, y)$. Then g is easily seen to be continuous and $\pi \circ g = \mathrm{id}_Y$ so the required result comes from question 5.

7. We take as our alphabet:

   A B C D E F G H I J K L M N O P Q R S T U V W X Y Z

   and then the separate homeomorphism classes are

   {A R} {B} {C G I J L M N S U V W Z} {D O} {E F T Y} {H K} {P} {Q} {X}