Chapter 1

Topology: Concepts and Examples

1.1 Revision

Definition. A *topological space* $(X, \mathcal{T})$ is a set X together with a distinguished family of subsets $\mathcal{T} \subset \mathcal{P}(X)$ of X, the *open subsets* of X, such that

(i) $\emptyset, X \in \mathcal{T}$;

(ii) if $\{G_\alpha\}_{\alpha \in I} \subset \mathcal{T}$ then $\bigcup_{\alpha \in I} G_\alpha \in \mathcal{T}$;

(iii) if $G_1, G_2 \in \mathcal{T}$ then $G_1 \cap G_2 \in \mathcal{T}$. It then follows, by induction, that if $G_1, \ldots, G_n \in \mathcal{T}$, then $\bigcap_{i=1}^n G_i \in \mathcal{T}$.

In this case, $\mathcal{T}$ is said to be a *topology* on X.

A subset $F \subset X$ is said to be *closed* if $X \setminus F \in \mathcal{T}$ (that is, if $X \setminus F$ is open).

For $x \in X$, a *neighbourhood* of x is a subset $A \subset X$ such that there exists $G \in \mathcal{T}$ with $x \in G \subset A$ (or in other words, if $x \in A^\circ$, the *interior of* A).

Definition. Let $(X, \mathcal{T})$ and $(Y, \mathcal{T}')$ be topological spaces. A map $f : X \to Y$ is said to be *continuous* if

$$f^{-1}(G) \in \mathcal{T}, \quad \forall G \in \mathcal{T}',$$

that is, if the inverse image of open sets are open, or, equivalently, the inverse image of closed sets are closed[1].

A map $f : X \to Y$ is said to be a *homeomorphism* if f is a continuous bijection with continuous inverse $f^{-1} : Y \to X$. In this case, X and Y are said to be *homeomorphic* and we write $X \cong Y$.

A map $f : X \to Y$ is said to be *open* if $f(G) \in \mathcal{T}'$, $\forall G \in \mathcal{T}$. Note that f is a homeomorphism if and only if f is continuous, open and bijective.

Definition. A *base* for a topology $\mathcal{T}$ on a set X is a collection of open subsets $\mathcal{B} \subset \mathcal{T}$ of X such that each non-empty open set in X is a union of elements of $\mathcal{B}$.

Proposition 1.1. *A family $\mathcal{B}$ of subsets of X is a base for some (necessarily unique) topology on X if and only if*

(i) $\bigcup_{B \in \mathcal{B}} B = X$;

(ii) if $x \in B_1 \cap B_2$, where $B_1, B_2 \in \mathcal{B}$, then $\exists B_3 \in \mathcal{B}$ such that $x \in B_3 \subset B_1 \cap B_2$.

[1] This latter condition is sometimes easier to check.

Definition. A subbase $\mathcal{S}$ for a topology $\mathcal{T}$ is a collection of open sets $\mathcal{S} \subset \mathcal{T}$ such that the collection of all finite intersections of elements of $\mathcal{S}$ forms a base.

Corollary 1.2. *Any collection $\mathcal{S}$ of subsets of a set X with union equal to X forms a subbase for some unique topology on X.*

Definition. Let $(X, \mathcal{T})$ be a topological space and let $A \subset X$. The induced topology $\mathcal{T}_A$ on A is defined by

$$\mathcal{T}_A = \{A \cap G : G \in \mathcal{T}\}.$$

Proposition 1.3. *Let $(X_1, \mathcal{T}_1), \ldots, (X_n, \mathcal{T}_n)$ be topological spaces. The product topology is the unique topology on $X_1 \times \cdots \times X_n$ with the property that, for any topological space Y and map $f : Y \to X_1 \times \cdots \times X_n$, f is continuous if and only if each component $\pi_i \circ f : Y \to X_i$ is continuous.*

Proposition 1.4. *Let $(Y, \mathcal{S}), (X_1, \mathcal{T}_1), (X_2, \mathcal{T}_2)$ be topological spaces and $f : X_1 \times X_2 \to Y$ be a continuous map, where $X_1 \times X_2$ has the product topology. For $x_1 \in X_1$, define $f_{x_1} : X_2 \to Y$ by $f_{x_1}(x_2) = f(x_1, x_2)$, $\forall x_2 \in X_2$, and similarly define $f_{x_2} : X_1 \to Y$ for $x_2 \in X_2$. Then f_{x_1} and f_{x_2} are continuous.*

Definition. Let $(X, \mathcal{T})$ be a topological space and $\pi : X \to Y$ a surjective map onto Y. The *quotient topology $\mathcal{T}_\pi$ on Y induced by π* is defined by

$$\mathcal{T}_\pi = \{G \subset Y : \pi^{-1}G \in \mathcal{T}\}.$$

Proposition 1.5. *Let X be a topological space and $\pi : X \to Y$ a surjection onto a set Y. The quotient topology $\mathcal{T}_\pi$ is the* unique *topology on Y with the property that, for all topological spaces Z and maps $f : Y \to Z$, f is continuous if and only if $f \circ \pi$ is continuous.*

1.2 Some serious examples

Definition. The *general linear group* is the subset $\mathrm{GL}(n, \mathbb{R}) \subset M(n)$ given by

$$\mathrm{GL}(n, \mathbb{R}) = \{A \in M(n) : \det A \neq 0\}.$$

Otherwise said, it is the set of invertible $n \times n$ matrices.

Definition. A *topological group* is a Hausdorff topological space $(G, \mathcal{T})$, such that G is a group and the functions $\mu : G \times G \to G$, $(a, b) \mapsto ab$ and $i : G \to G$, $a \mapsto a^{-1}$ are both continuous. (Here $G \times G$ has the product topology.)

Definition. The *orthogonal group* $\mathrm{O}(n)$ is defined by

$$\mathrm{O}(n) = \{A \in M(n) : AA^T = I\}.$$

The *special orthogonal group* $\mathrm{SO}(n)$ is defined by

$$\mathrm{SO}(n) = \{A \in \mathrm{O}(n) : \det A = 1\}.$$

Definition. The topological quotient $\mathbb{R}^{n+1} \setminus \{0\}/_\sim$, where $v \sim w$ if and only if $v = \lambda w$, for some $\lambda \in \mathbb{R}$, is called *n-dimensional real projective space* and denoted $\mathbb{R}P^n$.

Theorem 1.6. *$S^n/_{\sim'} \cong \mathbb{R}P^n$, where $v_1 \sim' v_2$ if and only if $v_1 = \pm v_2$.*

Chapter 2

Homotopy and the Fundamental Group

2.1 Paths and the Homotopy Relation

Definition. Let X be a topological space. A *path* in X is a continuous map $\gamma : [0,1] \to X$. In this case, $\gamma(0)$ and $\gamma(1) \in X$ are said to be the *end-points of* γ, and γ is said to *go from* $\gamma(0)$ *to* $\gamma(1)$.

Definition. A topological space $(X, \mathcal{T})$ is said to be *connected* if there do not exist $G_1, G_2 \in \mathcal{T}$ such that $X = G_1 \cup G_2$, $G_1 \cap G_2 = \emptyset$ and $G_1, G_2 \neq \emptyset$.

A topological space X is said to be *path-connected* if, for all $x_1, x_2 \in X$, there exists a path from x_1 to x_2 in X.

Definition. A topological space X is said to be *locally path-connected* if any neighbourhood of any point $x \in X$ contains a path-connected neighbourhood of x.

Definition. Let $\gamma_0, \gamma_1 : [0,1] \to X$ be paths with the same end-points: $(\gamma_0(0) = \gamma_1(0)$ and $\gamma_0(1) = \gamma_1(1))$. Say that γ_0 is *based homotopic to* γ_1 if there is a continuous map $F : [0,1] \times [0,1] \to X$ such that

$$\begin{aligned} F(t,0) &= \gamma_0(t), \quad t \in [0,1] \\ F(t,1) &= \gamma_1(t), \quad t \in [0,1] \\ F(0,s) &= \gamma_0(0) = \gamma_1(0), \quad s \in [0,1] \\ F(1,s) &= \gamma_0(1) = \gamma_1(1), \quad s \in [0,1] \end{aligned}$$

In this case, say that F is a *(based) homotopy from* γ_0 *to* γ_1 and write $\gamma_0 \sim \gamma_1$.

Lemma 2.1. *Let X be a topological space and A, B closed subsets of X such that $A \cup B = X$. Let $\phi : X \to Y$ be a map into a topological space Y such that $\phi_{|A}$ and $\phi_{|B}$ are continuous with respect to the induced topologies on A and B respectively. Then ϕ is continuous on X.*

Lemma 2.2. *The based homotopy relation $\sim$ is an equivalence relation.*

Definition. Let

$$\mathrm{Path}(X, x_0, y_0) = \{\gamma \in \mathcal{C}([0,1], X) \mid \gamma(0) = x_0, \gamma(1) = y_0\},$$

so that $\sim$ is an equivalence relation on $\mathrm{Path}(X, x_0, y_0)$. The set of equivalence classes (*based homotopy classes*) is denoted by $\pi_1(X, x_0, y_0)$.

Definition. Let $\alpha, \beta : [0,1] \to X$ be paths in X such that $\alpha(1) = \beta(0)$. The *product* or *join* $\alpha \cdot \beta : [0,1] \to X$ is defined by

$$(\alpha \cdot \beta)(t) = \begin{cases} \alpha(2t), & t \in [0, \frac{1}{2}] \\ \beta(2t-1), & t \in [\frac{1}{2}, 1]. \end{cases}$$

Theorem 2.3. *Let $\alpha_0, \alpha_1 : [0,1] \to X$ be paths from x_0 to y_0 in X and let $\beta_0, \beta_1 : [0,1] \to X$ be paths from y_0 to z_0 in X. If $\alpha_0 \sim \alpha_1$ and $\beta_0 \sim \beta_1$, then $\alpha_0 \cdot \beta_0 \sim \alpha_1 \cdot \beta_1$.*

2.2 The Fundamental Group

Notation. Henceforth, we denote the unit interval $[0,1]$ by I.

Lemma 2.4. *Let $[\alpha] \in \pi_1(X, x_0, y_0)$, $[\beta] \in \pi_1(X, y_0, z_0)$ and $[\gamma] \in \pi_1(X, z_0, u_0)$. Then*

$$([\alpha] \cdot [\beta]) \cdot [\gamma] = [\alpha] \cdot ([\beta] \cdot [\gamma]).$$

Notation. For $x \in X$, let $\gamma_x : I \to X$ denote the constant path given by $\gamma_x(t) = x$. Moreover, let $1_x = [\gamma_x] \in \pi_1(X, x, x)$.

Lemma 2.5. *Let $a \in \pi_1(X, x, y)$ and $b \in \pi_1(X, y, x)$. Then $1_x \cdot a = a$ and $b \cdot 1_x = b$.*

Lemma 2.6. *Let $a \in \pi_1(X, x, y)$ so that $\bar{a} \in \pi_1(X, y, x)$. Then $a \cdot \bar{a} = 1_x$ and $\bar{a} \cdot a = 1_y$.*

Notation. In view of the above, we write a^{-1} instead of $\bar{a}$, for $a \in \pi_1(X, x, y)$.

Corollary 2.7. *The operation $\cdot$ endows the set of based homotopy classes $\pi_1(X, x, x)$ with the structure of a group.*

Definition. A path $\gamma : I \to X$ is said to be a *loop* if $\gamma(0) = \gamma(1)$, and in this case γ is said to be *based at* $\gamma(0)$.

The set of all based homotopy classes of loops based at $x \in X$ is denoted by $\pi_1(X, x)$ (instead of $\pi_1(X, x, x)$) and is called *the fundamental group*[1] *of X based at x.*

Proposition 2.8. *Let $x, y \in X$, a path-connected topological space. Then $\pi_1(X, x) \cong \pi_1(X, y)$.*

2.3 Continuous Maps and the Fundamental Group

Proposition 2.9. *Let $\gamma_0, \gamma_1 : I \to X$ be paths in X and $\phi : X \to Y$ continuous.*

(i) If $\gamma_0 \sim \gamma_1$, then $\phi \circ \gamma_0 \sim \phi \circ \gamma_1$.

(ii) If $\gamma_0(1) = \gamma_1(0)$, then $\phi \circ (\gamma_0 \cdot \gamma_1) = (\phi \circ \gamma_0) \cdot (\phi \circ \gamma_1)$.

(iii) $\phi \circ \overline{\gamma}_0 = \overline{(\phi \circ \gamma_0)}$.

(iv) $\phi \circ \gamma_x = \gamma_{\phi(x)}$.

Corollary 2.10. *$\phi_* : \pi_1(X, x) \to \pi_1(Y, \phi(x))$ is a homomorphism of groups.*

Theorem 2.11. *If $\phi : X \to Y$ is a homeomorphism, then for all $x \in X$, the map $\phi_* : \pi_1(X, x) \to \pi_1(Y, \phi(x))$ is an isomorphism.*

Definition. Two continuous maps $\phi_0, \phi_1 : X \to Y$ of topological spaces are said to be *homotopic* if there exists a continuous map $F : X \times I \to Y$ (here $X \times I$ has the product topology) such that

$$F(x, 0) = \phi_0(x), \quad F(x, 1) = \phi_1(x), \quad \forall x \in X.$$

In this case, we write: $\phi_0 \simeq \phi_1$ and say that F is a *homotopy from ϕ_0 to ϕ_1*.

[1] $\pi_1(X, x)$ is also sometimes called the *Poincaré group* of X after its inventor Henri Poincaré (1854–1912).

Definition. Let $A \subset X$ and $\phi_0, \phi_1 : X \to Y$ be continuous maps. Then, ϕ_0 is said to be *homotopic to* ϕ_1 *relative to* A if there exists a continuous map $F : X \times I \to Y$ such that

$$\begin{aligned} F(x,0) &= \phi_0(x), \\ F(x,1) &= \phi_1(x), \\ F(a,s) &= \phi_0(a) = \phi_1(a), \end{aligned}$$

for all $x \in X$, $a \in A$, $s \in I$. (In particular, $\phi_{0|A} = \phi_{1|A}$.)

In this case, we write $\phi_0 \simeq \phi_1$ rel A and say that F is a *homotopy relative to* A *from* ϕ_0 *to* ϕ_1.

Theorem 2.12. *If* $\phi_0, \phi_1 : X \to Y$ *are homotopic relative to* $\{x\}$, *for some* $x \in X$, *then*

$$(\phi_0)_* = (\phi_1)_* : \pi_1(X,x) \to \pi_1(Y, \phi_0(x)) = \pi_1(Y, \phi_1(x)).$$

Lemma 2.13. *Let* $F : I \times I \to X$ *be continuous. Define paths* $\alpha, \beta, \gamma, \delta : I \to X$ *as in the diagram on the left. Thus*

$$\alpha(s) = F(0,s), \quad \beta(s) = F(1,s), \quad \gamma(t) = F(t,0), \quad \delta(t) = F(t,1),$$

for $t, s \in I$.

Then $\delta \sim \overline{\alpha} \cdot \gamma \cdot \beta$.

Theorem 2.14. *Let* $\phi_0, \phi_1 : X \to Y$ *be homotopic via a homotopy* $F : X \times I \to Y$ *and let* $x \in X$. *Let* $\gamma : I \to Y$ *be the path in* Y *from* $\phi_0(x)$ *to* $\phi_1(x)$ *given by* $\gamma(s) = F(x,s)$, *for* $s \in I$, *and set* $g = [\gamma]$.

Then $(\phi_1)_* = g_* \circ (\phi_0)_*$ *so that we have a commuting diagram:*

$$\begin{array}{ccc} \pi_1(X,x) & \xrightarrow{(\phi_0)_*} & \pi_1(Y,\phi_0(x)) \\ & \searrow^{(\phi_1)_*} & \downarrow g_* \\ & & \pi_1(Y,\phi_1(x)) \end{array}$$

Corollary 2.15. $(\phi_0)_* : \pi_1(X,x) \to \pi_1(Y, \phi_0(x))$ *is an isomorphism if and only if* $(\phi_1)_* : \pi_1(X,x) \to \pi_1(Y, \phi_1(x))$ *is an isomorphism.*

Definition. Two topological spaces X and Y are said to be *homotopy equivalent* (or *have the same homotopy type*) if there exist continuous maps $\phi : X \to Y$ and $\psi : Y \to X$ such that $\psi \circ \phi \simeq \mathrm{id}_X$ and $\phi \circ \psi \simeq \mathrm{id}_Y$.

In this case, we write $X \simeq Y$ and say that ϕ is a *homotopy equivalence* with *homotopy inverse* ψ.

Theorem 2.16. *Let* $\phi : X \to Y$ *be a homotopy equivalence. Then* $\phi_* : \pi_1(X,x) \to \pi_1(Y, \phi(x))$ *is an isomorphism for any* $x \in X$.

Thus homotopy-equivalent path-connected spaces have isomorphic fundamental groups.

Definition. Let X be a topological space and let $A \subset X$ with inclusion $i : A \to X$. Say that A is a *retract* of X if there exists a continuous map $r : X \to A$ such that $r(a) = a$, $\forall a \in A$, that is, $r \circ i = \mathrm{id}_A$. In this case, the map $r : X \to A$ is called a *retraction*.

A is said to be a *deformation retract* of X if there exists a retraction $r : X \to A$ such that $i \circ r \simeq \mathrm{id}_X$ rel A.

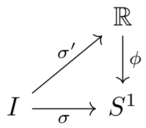
Definition. A topological space X is said to be *simply connected* if it is path-connected and $\pi_1(X) = \{1\}$.

X is said to be *contractible* if X is homotopy equivalent to a singleton set: $X \simeq \{\mathrm{pt}\}$.

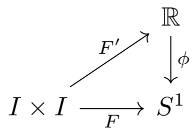
2.4 Calculating the Fundamental Group

Theorem 2.17. $\pi_1(S^1, 1) \cong \mathbb{Z}$.

Lemma 2.18 (Path-Lifting Lemma). *If $\sigma : I \to S^1$ is a path beginning at $1 \in S^1$, then there exists a* unique *path $\sigma' : I \to \mathbb{R}$ starting at 0 such that $\phi \circ \sigma' = \sigma$. Thus we have a commuting diagram:*



Lemma 2.19 (Homotopy-Lifting Lemma). *If $\tau : I \to S^1$ is another path beginning at 1, and $\sigma \sim \tau$ via a homotopy $F : I \times I \to S^1$, then $\sigma' \sim \tau'$ via a unique homotopy $F' : I \times I \to \mathbb{R}$ such that $\phi \circ F' = F$. Thus we have a commuting diagram:*



Corollary 2.20. *$\sigma'(1)$ depends only on the based homotopy class of σ.*

Lemma 2.21 (Lebesgue Covering Lemma). *Let (X, d) be a compact metric space and let $\{U_\alpha\}_{\alpha \in I}$ be an open cover of X. Then $\exists \delta > 0$ such that any subset of diameter strictly less than δ is contained in one of the sets U_α.*

Such a δ is said to be a Lebesgue number *of the cover.*

Theorem 2.22. *Let X be a topological space, and suppose that $X = U \cup V$ where $U, V \subset X$ are open and simply connected, while $U \cap V$ is non-empty and path-connected. Then X is simply connected.*

Corollary 2.23. *The n-sphere S^n is simply connected, for $n \geq 2$.*

Theorem 2.24. *Let X and Y be topological spaces and, as usual, equip $X \times Y$ with the product topology. Then*

$$\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0),$$

for any $(x_0, y_0) \in X \times Y$.

Corollary 2.25. *$\pi_1(T^2) \cong \mathbb{Z} \times \mathbb{Z}$.*

2.5 Two Applications

Fundamental Theorem of Algebra. *A non-constant polynomial with complex coefficients has a complex root.*

Lemma 2.26. *Let $A \subseteq X$ be a retract of X with retraction $r : X \to A$. Then, $r_* : \pi_1(X, a) \to \pi_1(A, a)$ is surjective, for any $a \in A$.*

Brouwer Fixed Point Theorem. *Let $D^n = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$ be the closed unit disc*[2] *in $\mathbb{R}^n$ and let $f : D^n \to D^n$ be continuous.*

Then f has a fixed point: there is $x_0 \in D^n$ such that $f(x_0) = x_0$.

[2] Here we use the Euclidean norm but, in fact, any norm will do.

Chapter 3

Covering Spaces

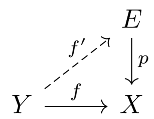
3.1 Covering spaces and lifting theorems

Definition. A *covering map* $p : E \to X$ is a continuous surjection such that each $x \in X$ has an open neighbourhood U for which $p^{-1}(U)$ is a union of *disjoint* open sets S_i, $i \in I$, with $p|_{S_i} : S_i \to U$ a homeomorphism for each $i \in I$.

In this case, each such set U is said to be *evenly covered*, and the S_i, $i \in I$, are called the *sheets* over U.

A topological space E is said to be a *covering space* of X if there exists a covering map $p : E \to X$.

Definition. Let $p : E \to X$ be a covering map and $f : Y \to X$ a continuous map of topological spaces. A *lift of f (with respect to p)* is a continuous map $f' : Y \to E$ such that $p \circ f' = f$. Thus f' makes the following diagram commute:



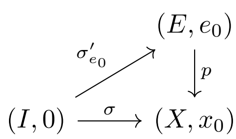
Lemma 3.1 (Unique Lifting Property)**.** *Let Y be connected, $p : E \to X$ be a covering map, and $f : Y \to X$ continuous. Suppose that $f_1, f_2 : Y \to E$ are two lifts of f, that is, $p \circ f_1 = p \circ f_2 = f$. Then*

$$A := \{y \in Y : f_1(y) = f_2(y)\}$$

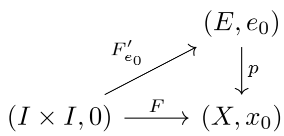
is either $\emptyset$ or Y.

Notation. Write $f : (X, x) \to (Y, y)$ if $f : X \to Y$, $x \in X$, $y \in Y$ and $f(x) = y$.

Theorem 3.2 (Path-Lifting Theorem)**.** *Let $p : (E, e_0) \to (X, x_0)$ be a covering map and $\sigma : (I, 0) \to (X, x_0)$ a path, starting at $x_0 \in X$. Then there exists a unique lift $\sigma'_{e_0} : (I, 0) \to (E, e_0)$:*



Theorem 3.3 (Homotopy Lifting I)**.** *Let $p : (E, e_0) \to (X, x_0)$ be a covering map and $F : (I \times I, 0) \to (X, x_0)$ a continuous map. Then there is a unique lift $F'_{e_0} : (I \times I, 0) \to (E, e_0)$:*



Corollary 3.4 (Homotopy Lifting II)**.** *Let $p : (E, e_0) \to (X, x_0)$ be a covering map. If $\sigma, \tau : (I, 0) \to (X, x_0)$ are continuous and $\sigma \sim \tau$, then $\sigma'_{e_0} \sim \tau'_{e_0}$.*

In particular, $\sigma'_{e_0}(1) = \tau'_{e_0}(1)$.

Corollary 3.5. *Let $p : (E, e_0) \to (X, x_0)$ be a covering map. Then $p_* : \pi_1(E, e_0) \to \pi_1(X, x_0)$ injects.*

Definition. A *(right) action of a group G on a set A* is a map $A \times G \to A$, written $(a, g) \mapsto a \cdot g$, such that

$$\begin{aligned} a \cdot 1 &= a, \\ a \cdot (g_1 g_2) &= (a \cdot g_1) \cdot g_2, \end{aligned}$$

for all $a \in A$, $g_1, g_2 \in G$.

Theorem 3.6 (Ultimate Lifting Theorem)**.** *Let $p : (E, e_0) \to (X, x_0)$ be a covering map, Y a connected and locally path-connected topological space and $f : (Y, y_0) \to (X, x_0)$ a continuous map. Then f has a unique*[1] *lift $f' : (Y, y_0) \to (E, e_0)$ if and only if*

$$f_*(\pi_1(Y, y_0)) \subset p_*(\pi_1(E, e_0)).$$

Corollary 3.7. *If Y is simply connected and locally path-connected, then any continuous function $f : (Y, y_0) \to (X, x_0)$ has a lift $f' : (Y, y_0) \to (E, e_0)$.*

3.2 The Fundamental Group and Deck Translations

Definition. Let $p : (E, e_0) \to (X, x_0)$ be a covering map. A *deck translation of p* is a homeomorphism $\phi : E \to E$ such that $p \circ \phi = p$. Thus:

$$\begin{array}{ccc} E & \xrightarrow{\phi} & E \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle p} & \\ & X & \end{array}$$

Theorem 3.8. *Let E be simply connected and locally path-connected and $p : (E, e_0) \to (X, x_0)$ a covering map. Let G be the group of deck translations of p. Then $G \cong \pi_1(X, x_0)$.*

3.3 Universal Covers

Definition. A topological space E is a *universal cover of X* if E is simply connected and there is a covering map $p : E \to X$.

Proposition 3.9 (Uniqueness of Universal Covers)**.** *If $p : E \to X$ and $\tilde{p} : \tilde{E} \to X$ are both universal covers of a locally path-connected space X, then there is a homeomorphism $\phi : E \to \tilde{E}$ such that $\tilde{p} \circ \phi = p$. Thus:*

$$\begin{array}{ccc} E & \xrightarrow{\phi} & \tilde{E} \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle \tilde{p}} & \\ & X & \end{array}$$

Definition. A topological space X is said to be *semi-locally simply connected* if each point $x \in X$ has a neighbourhood U with the property that any loop in U is based homotopic in X to the constant loop.

[1] By Theorem 3.1!

3.4 Topology of $\mathbb{R}P^n$ and the Borsuk–Ulam Theorem

Theorem 3.10.

1. $\pi_1(\mathbb{R}P^1) \cong \mathbb{Z}$.
2. $\pi_1(\mathbb{R}P^n) \cong \mathbb{Z}/2\mathbb{Z}$, *for* $n \geq 2$.

Theorem 3.11 (Borsuk–Ulam)**.** *There does not exist a continuous map* $\phi : S^n \to S^{n-1}$ *which is* antipode preserving, *that is, such that* $\phi(-x) = -\phi(x)$, *for all* $x \in S^n$.

Corollary 3.12. *If* $f : S^n \to \mathbb{R}^n$ *is continuous and* odd, *that is,* $f(-x) = -f(x)$, *for all* $x \in S^n$, *then there is* $x_0 \in S^n$ *such that* $f(x_0) = 0$.

Corollary 3.13. *If* $f : S^n \to \mathbb{R}^n$ *is continuous, then there is* $x_0 \in S^n$ *such that* $f(x_0) = f(-x_0)$.

(In particular, f *does not inject.)*

Theorem 3.14 (Ham Sandwich Theorem)**.** *Let* $X_1, \ldots, X_n$ *be bounded Lebesgue measurable subsets of* $\mathbb{R}^n$*. Then there exists a hyperplane bisecting each set* X_i *simultaneously: that is, each* X_i *has the same volume*[2] *on each side of the hyperplane.*

[2] We really mean the same Lebesgue measure and so area when $n = 2$.