MA40040: Algebraic Topology

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Introduction

The name of the game

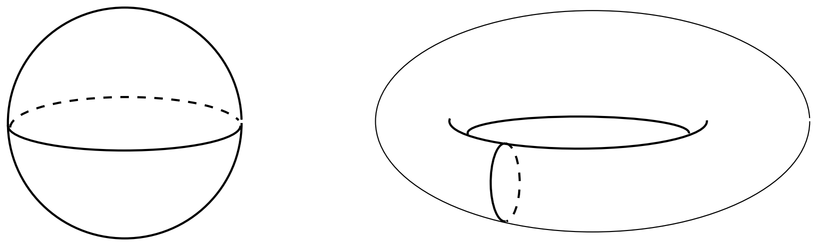
What is *algebraic topology*? Let us begin by thinking about the ingredients:

Topology is the part of analysis and geometry that is concerned with open sets and the concepts that are derived from them such as compactness, connectedness and separation along with continuous functions and maps.

Algebra is concerned with sets equipped with binary and other operations such as groups; vector spaces; rings and fields, along with the structure-preserving maps between these sets such as group homomorphisms; linear transformations and so on.

Algebraic topology is not so much a mixture of these topics[1] as a family of methods to reduce (difficult) problems in topology to (possibly easier) problems in algebra. Invented by Poincaré, this is one of the Big Ideas of 20th century mathematics.

Here is an example of the sort of thing I mean. Contemplate the 2-sphere $S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\} \subset \mathbb{R}^3$ and the 2-torus $T^2 = \{(x, y, z) \in \mathbb{R}^3 : ((x^2 + y^2)^{1/2} - 2)^2 + z^2 = 1\} \subset \mathbb{R}^3$.



Equip both sets with the induced topology from the Euclidean topology on $\mathbb{R}^3$ and ask:

Are the 2*-sphere and the* 2*-torus homeomorphic topological spaces?*

Intuitively, the answer is clearly no: the torus has a great big hole in it! A more challenging question is how would one *prove* that these spaces are not homeomorphic?

Here is a possible strategy: suppose that we had a way to associate a group $G(T)$ to each topological space T in such a way that if T_1 and T_2 are homeomorphic topological spaces, then $G(T_1)$ and $G(T_2)$ are isomorphic groups. We could then compute the groups $G(S^2)$ and $G(T^2)$ and try to see if they were isomorphic. If they are not, then we have proved that S^2 and T^2 are not homeomorphic! In this way, we have converted a problem in topology to a problem in algebra, in fact, group theory.

To construct such a G is the main mission of this course. It will turn out that, for the G we construct, $G(S^2) = \{1\}$, the trivial group, while $G(T^2) \cong \mathbb{Z} \times \mathbb{Z}$. We do not need to be experts in Group Theory to see that these groups are not isomorphic!

[1] Although such mixtures exist and are interesting: see the brief discussion of topological groups below.

Remark. There is another approach to our question that you may be familiar with: a compact surface S has an *Euler characteristic* $\chi(S)$ and homeomorphic surfaces have the same Euler characteristic. So we are done when we note that $\chi(S^2) = 2 \neq 0 = \chi(T^2)$.

However, to make this argument rigorous takes a lot of work since it relies on the existence of triangulations (a deep result of itself) and the fact that the Euler characteristic is independent of the choice of triangulation.

Applications

Our motivating problem came from Topology but Algebraic Topology has applications all over mathematics and some of them are quite startling. Here are some of the amazing theorems we shall prove:

The Fundamental Theorem of Algebra Every non-constant polynomial with complex coefficients has a complex root. You have known this result since you were at school and engineers use it every day but proofs are not easy to come by. There is a Complex Analysis proof: if p is a never-vanishing polynomial, then you can easily show that $1/p$ is bounded and complex analytic on all of $\mathbb{C}$ and so constant by Liouville's Theorem. There is also a proof by elementary analysis and Galois Theory: Galois Theory reduces things to the case where p has real coefficients and odd degree but such a polynomial has a real root by the Intermediate Value Theorem.

We shall give a proof which is more elementary (in the sense of requiring less background) than either of these by exploiting the non-trivial topology of the circle.

The Brouwer Fixed Point Theorem You all know the Contraction Mapping Principle (otherwise known as the Banach Fixed Point Theorem): a contraction mapping on a complete metric space has a fixed point. This is the key ingredient in the Picard Theorem on existence of solutions of ordinary differential equations. The Brouwer Fixed Point Theorem asserts that *any* continuous map of a closed ball in $\mathbb{R}^n$ has a fixed point. It has many fun corollaries and is the start of a family of ideas, *Degree Theory*, that can be used to prove existence of solutions of *partial differential equations*.

The Ham Sandwich Theorem Take two pieces of bread and one piece of ham. Place them anywhere you like in $\mathbb{R}^3$. Then there is a hyperplane that bisects each of the three ingredients (so that if you slide a, possibly very long, knife along this hyperplane, you will cut your sandwich in half!).

We will prove this using the non-trivial topology of the real projective plane.

Course outline

The course consists of three chapters:

Topology: Concepts and Examples Here we revise the key concepts of point-set topology that we will need and explore them via some substantial examples. Some of these examples will come back to haunt us later in the course.

Homotopy and the Fundamental Group We make the basic construction of the course and construct the *fundamental group* of a topological space. To do so, we will learn how to deform paths and other continuous maps. We will compute the fundamental group of several well-known spaces and, in particular, discover that the fundamental group of the circle is a copy of the integers $\mathbb{Z}$. This will enable us to prove the Fundamental Theorem of Algebra and the Brouwer Fixed Point Theorem.

Covering Spaces We will develop an alternative, perhaps more geometric, approach to the fundamental group that realises it as the symmetries of something. Fruits of this analysis will include the computation of the fundamental group of the real projective plane from which we will eventually deduce the Ham Sandwich Theorem.

Chapter 1

Topology: Concepts and Examples

1.1 Revision

Recall:

Definition. A *topological space* $(X, \mathcal{T})$ is a set X together with a distinguished family of subsets $\mathcal{T} \subset \mathcal{P}(X)$ of X, the *open subsets* of X, such that

(i) $\emptyset, X \in \mathcal{T}$;

(ii) if $\{G_\alpha\}_{\alpha \in I} \subset \mathcal{T}$ then $\bigcup_{\alpha \in I} G_\alpha \in \mathcal{T}$;

(iii) if $G_1, G_2 \in \mathcal{T}$ then $G_1 \cap G_2 \in \mathcal{T}$. It then follows, by induction, that if $G_1, \ldots, G_n \in \mathcal{T}$, then $\bigcap_{i=1}^n G_i \in \mathcal{T}$.

In this case, $\mathcal{T}$ is said to be a *topology* on X.

A subset $F \subset X$ is said to be *closed* if $X \setminus F \in \mathcal{T}$ (that is, if $X \setminus F$ is open).

For $x \in X$, a *neighbourhood* of x is a subset $A \subset X$ such that there exists $G \in \mathcal{T}$ with $x \in G \subset A$ (or in other words, if $x \in A^\circ$, the *interior of* A).

We remark that a given set X admits many different topologies including the *discrete topology* where $\mathcal{T} = \mathcal{P}(X)$ and the *indiscrete topology* where $\mathcal{T} = \{\emptyset, X\}$. Despite this, we will frequently abuse notation and refer to X as a topological space if the topology in question is clear from the context.

Maps

Definition. Let $(X, \mathcal{T})$ and $(Y, \mathcal{T}')$ be topological spaces. A map $f : X \to Y$ is said to be *continuous* if

$$f^{-1}(G) \in \mathcal{T}, \quad \forall G \in \mathcal{T}',$$

that is, if the inverse image of open sets are open, or, equivalently, the inverse image of closed sets are closed[1].

A map $f : X \to Y$ is said to be a *homeomorphism* if f is a continuous bijection with continuous inverse $f^{-1} : Y \to X$. In this case, X and Y are said to be *homeomorphic* and we write $X \cong Y$.

A map $f : X \to Y$ is said to be *open* if $f(G) \in \mathcal{T}'$, $\forall G \in \mathcal{T}$. Note that f is a homeomorphism if and only if f is continuous, open and bijective.

[1] This latter condition is sometimes easier to check.

Here are some examples:

1. The identity map $\mathrm{id}_X : X \to X$, $x \mapsto x$, is continuous for any topology on X.
2. If $\phi_1 : X \to Y$ and $\phi_2 : Y \to Z$ are continuous maps of topological spaces, then $\phi_2 \circ \phi_1 : X \to Z$ is also continuous.
3. Constant maps of topological spaces are always continuous.

Remark. The first two conditions tell us that the collection of topological spaces and continuous maps constitute a *category*. We shall say more about this below.

Bases and Subbases

Definition. A *base* for a topology $\mathcal{T}$ on a set X is a collection of open subsets $\mathcal{B} \subset \mathcal{T}$ of X such that each non-empty open set in X is a union of elements of $\mathcal{B}$.

Example. Let (M, d) be a metric space. A base for the metric topology is given by

$$\mathcal{B} = \{B_\epsilon(x) : x \in M, \epsilon > 0\}.$$

Proposition 1.1. *A family $\mathcal{B}$ of subsets of X is a base for some (necessarily unique) topology on X if and only if*

(i) $\bigcup_{B \in \mathcal{B}} B = X$;

(ii) *if* $x \in B_1 \cap B_2$, *where* $B_1, B_2 \in \mathcal{B}$, *then* $\exists B_3 \in \mathcal{B}$ *such that* $x \in B_3 \subset B_1 \cap B_2$.

Proof. If $\mathcal{B}$ is a base then (i) holds because $X \in \mathcal{X}$ while (ii) holds because $B_1 \cap B_2 \in \mathcal{X}$.

For the converse, given $\mathcal{B}$ with properties (i) and (ii), set

$$\mathcal{T} = \{\text{all unions of elements of } \mathcal{B}\}$$

and prove that this satisfies the axioms of a topology. □

Definition. A subbase $\mathcal{S}$ for a topology $\mathcal{T}$ is a collection of open sets $\mathcal{S} \subset \mathcal{T}$ such that the collection of all finite intersections of elements of $\mathcal{S}$ forms a base.

Corollary 1.2. *Any collection $\mathcal{S}$ of subsets of a set X with union equal to X forms a subbase for some unique topology on X.*

Proof. Set

$$\mathcal{B} = \{S_{i_1} \cap \cdots \cap S_{i_k} : S_{i_j} \in \mathcal{S}, k \in \mathbb{N}\},$$

and use Proposition 1.1. □

Examples.

1. The collection of intervals
$$\{(a, b) : a < b, a, b \in \mathbb{R}\}$$
is a base for the metric topology on $\mathbb{R}$.
2. The collection of intervals
$$\{(a, \infty) : a \in \mathbb{R}\} \cup \{(-\infty, b) : b \in \mathbb{R}\}$$
is a subbase for the same topology.

Three constructions

Induced topology

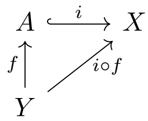
Definition. Let $(X, \mathcal{T})$ be a topological space and let $A \subset X$. The induced topology $\mathcal{T}_A$ on A is defined by

$$\mathcal{T}_A = \{A \cap G : G \in \mathcal{T}\}.$$

The induced topology is characterised by a *universal property* which is frequently easier to work with than the original definition:

Exercise. Let $i : A \hookrightarrow X$ be the inclusion. Then $\mathcal{T}_A$ is the *unique* topology on A such that, for any topological space Y and map $f : Y \to A$, f is continuous if and only if $i \circ f$ is continuous.

Thus, in the commuting diagram



one upward pointing arrow is continuous if and only if the other is.

Product topology

Let $(X_1, \mathcal{T}_1), \ldots, (X_n, \mathcal{T}_n)$ be topological spaces and contemplate the Cartesian product $X_1 \times \cdots \times X_n = \{(x_1, \ldots, x_n) : x_i \in X_i\}$. We have *projection maps* $\pi_i : X_1 \times \cdots \times X_n \to X_i$ given by

$$\pi_i(x_1, \ldots, x_n) = x_i.$$

Define $\mathcal{B} \subset \mathcal{P}(X_1 \times \cdots \times X_n)$ by

$$\{G_1 \times \cdots \times G_n : G_i \in \mathcal{T}_i\}$$

and observe that Proposition 1.1 applies to show that this is a base for a topology on $X_1 \times \cdots \times X_n$ called the *product topology*.

Warning: $\mathcal{B}$ is not usually a topology in its own right: it is not even closed under finite unions. Thus, a typical open set in the product topology is a *union* of sets in $\mathcal{B}$.

Again, the essential properties of the product topology are captured by its universal property:

Proposition 1.3. *Let $(X_1, \mathcal{T}_1), \ldots, (X_n, \mathcal{T}_n)$ be topological spaces. The product topology is the unique topology on $X_1 \times \cdots \times X_n$ with the property that, for any topological space Y and map $f : Y \to X_1 \times \cdots \times X_n$, f is continuous if and only if each component $\pi_i \circ f : Y \to X_i$ is continuous.*

What about maps *from* a product?

Proposition 1.4. *Let $(Y, \mathcal{S}), (X_1, \mathcal{T}_1), (X_2, \mathcal{T}_2)$ be topological spaces and $f : X_1 \times X_2 \to Y$ be a continuous map, where $X_1 \times X_2$ has the product topology. For $x_1 \in X_1$, define $f_{x_1} : X_2 \to Y$ by $f_{x_1}(x_2) = f(x_1, x_2)$, $\forall x_2 \in X_2$, and similarly define $f_{x_2} : X_1 \to Y$ for $x_2 \in X_2$. Then f_{x_1} and f_{x_2} are continuous.*

Proof. Define $i_{x_1} : X_2 \to X_1 \times X_2$ by $x_2 \mapsto (x_1, x_2)$. Then i_{x_1} is continuous by Proposition 1.3 (since the first component is constant and the second component is the identity map on X_2) and $f_{x_1} = f \circ i_{x_1}$ is hence a composition of continuous maps. □

Quotient topology

Definition. Let $(X, \mathcal{T})$ be a topological space and $\pi : X \to Y$ a surjective map onto Y. The *quotient topology* $\mathcal{T}_\pi$ *on* Y *induced by* π is defined by

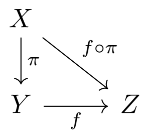
$$\mathcal{T}_\pi = \{G \subset Y : \pi^{-1}G \in \mathcal{T}\}.$$

Thus $G \subset Y$ is open if and only if $\pi^{-1}G$ is open and, in particular, π is continuous.

Once more, the quotient topology has a convenient characterisation by a universal property:

Proposition 1.5. *Let X be a topological space and $\pi : X \to Y$ a surjection onto a set Y. The quotient topology $\mathcal{T}_\pi$ is the* unique *topology on Y with the property that, for all topological spaces Z and maps $f : Y \to Z$, f is continuous if and only if $f \circ \pi$ is continuous.*

Thus, in the commuting diagram



one right-pointing arrow is continuous if and only if the other one is.

Remark. It is interesting to note that the universal properties for the induced and quotient topologies are very similar: swap π and i and change the direction of the arrows in the commuting diagrams to get from one to the other!

Here is an equivalent formulation that we will find useful: let $(X, \mathcal{T})$ be a topological space and $\sim$ an equivalence relation on X. Let $X/_\sim$ be the set of equivalence classes and $\pi : X \to X/_\sim$ the natural surjection $x \mapsto [x]$, sending $x \in X$ to its equivalence class. Give $X/_\sim$ the quotient topology $\mathcal{T}_\pi$ and call this the *topological quotient of X by $\sim$*.

Up to homeomorphism, this is equivalent to our previous formulation: given a surjection $\pi : X \to Y$, define an equivalence relation by $x \sim y$ if and only if $\pi(x) = \pi(y)$. Then $[x] \mapsto \pi(x)$ is a homeomorphism $X/_\sim \to Y$.

Examples.

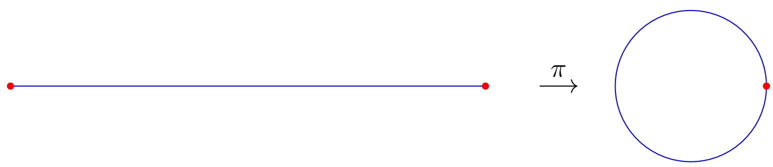
(i) Let $X = \mathbb{R}$ be equipped with the metric topology, and consider the relation $x \sim y \iff xy > 0$ or $x = y = 0$. Then,

$$X/_\sim = \{[-1], [0], [1]\}$$

and the open sets of $X/_\sim$ are $\emptyset, \{[-1]\}, \{[1]\}, \{[-1], [1]\}$ and $X/_\sim$. In particular, $X/_\sim$ is not Hausdorff since the only open set which contains $[0]$ is $X/_\sim$. *So a really well-behaved topological space can have a quite pathological quotient!*

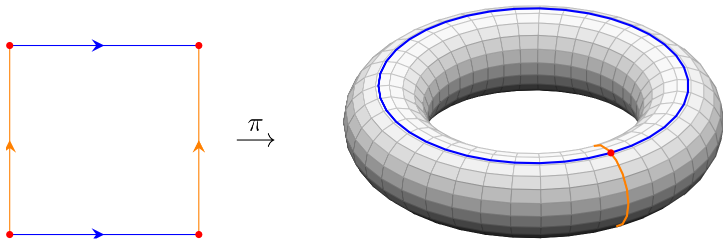
(ii) Let $X = [0, 2\pi] \subset \mathbb{R}$, equipped with the metric topology, and define a relation $\sim$ on X by $s \sim t$ if and only if $s = t$ or $\{s, t\} = \{0, 2\pi\}$. Then $X/_\sim \cong S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$. Indeed, the map $f : X/_\sim \to S^1$ given by $[t] \mapsto (\cos t, \sin t)$ is a well-defined bijection. Moreover $f \circ \pi : X \to S^1$ is given by $t \mapsto (\cos t, \sin t)$ and so is continuous. Thus f is continuous by Proposition 1.5. Moreover, $X/_\sim$ is compact being the continuous image (by π) of compact X while S^1 is Hausdorff. It follows that f is a homeomorphism (see exercise sheet 1).

The intuition here is that $X/_\sim$ is obtained from X by identifying equivalent points. In the case at hand, we start with an interval and "glue together" (thus identify) the endpoints to get a circle:

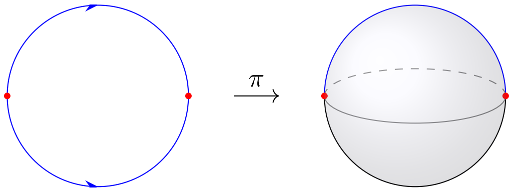


(iii) We know from M55 that any compact surface is a topological quotient of a polygon with certain edges identified. Examples:

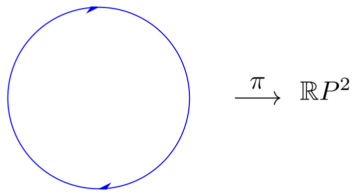
- T^2 is the quotient of a square with opposite edges identified:



- S^2 is the quotient of the unit disc where we identify (x, y) with $(x, -y)$ on the boundary circle:



- $\mathbb{R}P^2$ is the quotient of a disc with antipodal points identified:



1.2 Some serious examples

1.2.1 Topological Groups

Let $M(n)$ denote the set of $n \times n$ matrices with real entries. This is just a copy of $\mathbb{R}^{n^2}$ via

$$(a_{ij}) \mapsto (a_{11}, \dots, a_{1n}, a_{21}, \dots, a_{2n}, \dots, a_{n1}, \dots, a_{nn}),$$

and so we may equip $M(n)$ with the usual metric topology. Thus a sequence of matrices $(A^{(k)})_k$ converges to a matrix A if and only if each sequence of entries converges: $A^{(k)}_{ij} \to A_{ij}$, for all $1 \leq i, j \leq n$.

Definition. The *general linear group* is the subset $\mathrm{GL}(n, \mathbb{R}) \subset M(n)$ given by

$$\mathrm{GL}(n, \mathbb{R}) = \{A \in M(n) : \det A \neq 0\}.$$

Otherwise said, it is the set of invertible $n \times n$ matrices.

We know that $\mathrm{GL}(n, \mathbb{R})$ is a group under matrix multiplication but what about its topological properties?

First note that $\det : M(n) \to \mathbb{R}$ is a continuous function since $\det(A)$ is built of the entries of A using nothing but additions and multiplications (that is, it is *polynomial* in the entries of A). Thus, since

$$\mathrm{GL}(n, \mathbb{R}) = \det{}^{-1}(\mathbb{R} \setminus \{0\}),$$

we see that $\mathrm{GL}(n, \mathbb{R})$ is an open subset of $M(n)$ being the inverse image of the open set $\mathbb{R} \setminus \{0\}$ by a continuous map.

Now define $\mu : \mathrm{GL}(n,\mathbb{R}) \times \mathrm{GL}(n,\mathbb{R}) \to \mathrm{GL}(n,\mathbb{R})$ to be the group law (thus matrix multiplication): $\mu(A,B) = AB$ and $i : \mathrm{GL}(n,\mathbb{R}) \to \mathrm{GL}(n,\mathbb{R})$ by $i(A) = A^{-1}$. The maps μ and i encapsulate the group structure on $\mathrm{GL}(n,\mathbb{R})$. Observe that both these maps are continuous: indeed $(AB)_{ij}$ is polynomial in the entries of A and B while A^{-1}_{ij} is *rational* (that is, a quotient of polynomials) in the entries of A. Otherwise said, $GL(n,\mathbb{R})$ is a *topological group*:

Definition. A *topological group* is a Hausdorff topological space $(G,\mathcal{T})$, such that G is a group and the functions $\mu : G \times G \to G$, $(a,b) \mapsto ab$ and $i : G \to G$, $a \mapsto a^{-1}$ are both continuous. (Here $G \times G$ has the product topology.)

Here are two more topological groups:

Definition. The *orthogonal group* $\mathrm{O}(n)$ is defined by

$$\mathrm{O}(n) = \{A \in M(n) : AA^T = I\}.$$

The *special orthogonal group* $\mathrm{SO}(n)$ is defined by

$$\mathrm{SO}(n) = \{A \in \mathrm{O}(n) : \det A = 1\}.$$

Exercises.

1. $\mathrm{SO}(n)$ and $\mathrm{O}(n)$ are both subgroups of $\mathrm{GL}(n,\mathbb{R})$.
2. $\mathrm{SO}(n)$ and $\mathrm{O}(n)$ are compact subsets of $M(n)$.

1.2.2 The Compact-Open Topology

Let $(X,\mathcal{T})$ and $(Y,\mathcal{S})$ be topological spaces, and define

$$\mathcal{C}(X,Y) = \{\phi : X \to Y \mid \phi \text{ is continuous}\}.$$

For a compact set $K \subset X$ and an open set $G \subset Y$, set

$$\mathcal{C}_{K,G} = \{\phi \in \mathcal{C}(X,Y) \mid \phi(K) \subset G\}.$$

Note that

$$\mathcal{C}(X,Y) = \bigcup_{\substack{K\subset X \text{ compact} \\ G\subset Y \text{ open}}} \mathcal{C}_{K,G}.$$

Indeed, $\mathcal{C}(X,Y) = \mathcal{C}_{\{p\},Y}$ for any $p \in X$ (note that $\{p\}$ is finite and therefore compact in any topology). Hence, by Corollary 1.2, $\{\mathcal{C}_{K,G} : K \subset X \text{ compact}, G \subset Y \text{ open}\}$ is a subbase for a topology on $\mathcal{C}(X,Y)$, called the *compact-open topology*.

Example. Set $X = [a,b] \subset \mathbb{R}$ and $Y = \mathbb{R}$, both equipped with the usual metric topology. It follows that $\mathcal{C}(X,Y)$ is a normed linear space with norm

$$\|f\| = \sup_{x\in[a,b]} |f(x)|,$$

so that $\mathcal{C}(X,Y)$ is a metric space and therefore a topological space.

Exercise (Somewhat delicate)**.** The compact-open topology on $\mathcal{C}([a,b],\mathbb{R})$ coincides with the metric topology on $\mathcal{C}([a,b],\mathbb{R})$.

1.2.3 Real Projective Spaces

Define a relation $\sim$ on $\mathbb{R}^{n+1} \setminus \{0\}$ by $v \sim w \iff v = \lambda w$ for some $\lambda \in \mathbb{R}$, which is equivalent to saying that v and w lie on the same line through the origin. Then $\sim$ is an equivalence relation on $\mathbb{R}^{n+1} \setminus \{0\}$, and the equivalence classes are "punctured" lines through the origin (but not including the origin).

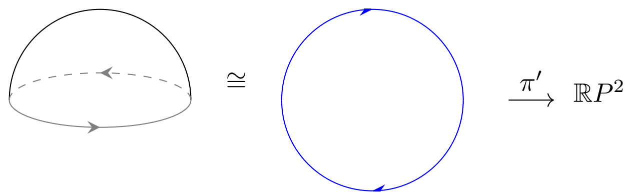
Definition. The topological quotient $\mathbb{R}^{n+1} \setminus \{0\}/_\sim$, where $v \sim w$ if and only if $v = \lambda w$, for some $\lambda \in \mathbb{R}$, is called *n-dimensional real projective space* and denoted $\mathbb{R}P^n$.

Here is another useful model for $\mathbb{R}P^n$: observe that each equivalence class in $\mathbb{R}P^n$ intersects the sphere S^n in exactly two antipodal points. So define an equivalence relation $\sim'$ on S^n by $v_1 \sim' v_2$ if and only if $v_1 = \pm v_2$. Note that for $v_1, v_2 \in S^n$, $v_1 \sim v_2$ if and only if $v_1 \sim' v_2$.

Theorem 1.6. *$S^n/_{\sim'} \cong \mathbb{R}P^n$, where $v_1 \sim' v_2$ if and only if $v_1 = \pm v_2$.*

Proof. Define $\phi : S^n/_{\sim'} \to \mathbb{R}P^n$ by $[v]' \mapsto [v]$, $\forall v \in S^n$. This is a well-defined bijection with inverse $[x] \mapsto [x/\|x\|]'$. Judicious use of Proposition 1.5 readily establishes continuity of both maps. $\square$

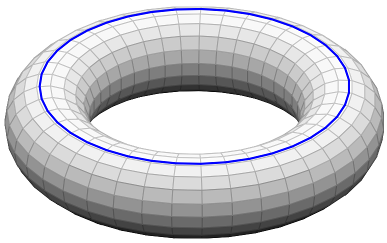
Example. Consider the case $n = 2$. Restrict the map π' to the closed upper hemisphere of S^2 to get a homeomorphism of $\mathbb{R}P^2$ with the upper hemisphere with antipodal points on the equator identified or, equivalently, the closed disc with boundary antipodal points identified:



Chapter 2

Homotopy and the Fundamental Group

Recall the problem of distinguishing the compact surfaces S^2 and T^2. Here is an idea: a closed loop on S^2 may always be deformed to the constant loop but this fails on T^2: the blue loop shown below cannot be so deformed because it goes around the hole in the torus.



This argument, made rigorous, will enable us to distinguish the two spaces.

The first step is therefore to make sense of "deforming" a path.

2.1 Paths and the Homotopy Relation

Definition. Let X be a topological space. A *path* in X is a continuous map $\gamma : [0,1] \to X$. In this case, $\gamma(0)$ and $\gamma(1) \in X$ are said to be the *end-points of* γ, and γ is said to *go from* $\gamma(0)$ *to* $\gamma(1)$.

We have met paths before, mainly in discussions of connectedness. Let us briefly recall that story.

Definition. A topological space $(X, \mathcal{T})$ is said to be *connected* if there do not exist $G_1, G_2 \in \mathcal{T}$ such that $X = G_1 \cup G_2$, $G_1 \cap G_2 = \emptyset$ and $G_1, G_2 \neq \emptyset$.

A topological space X is said to be *path-connected* if, for all $x_1, x_2 \in X$, there exists a path from x_1 to x_2 in X.

A path-connected topological space is always connected, but the converse may fail (remember the Topologist's Sine Curve!). The missing ingredient between connectedness and path-connectedness is *local path-connectedness*:

Definition. A topological space X is said to be *locally path-connected* if any neighbourhood of any point $x \in X$ contains a path-connected neighbourhood of x.

Exercises.

1. $\mathbb{R}^n$ (with the usual metric topology) is locally path-connected.

2. If X is connected and locally path-connected, then X is also path-connected.

Now for the first major definition in our programme:

Definition. Let $\gamma_0, \gamma_1 : [0,1] \to X$ be paths with the same end-points: ($\gamma_0(0) = \gamma_1(0)$ and $\gamma_0(1) = \gamma_1(1)$). Say that γ_0 is *based homotopic to* γ_1 if there is a continuous map $F : [0,1] \times [0,1] \to X$ such that

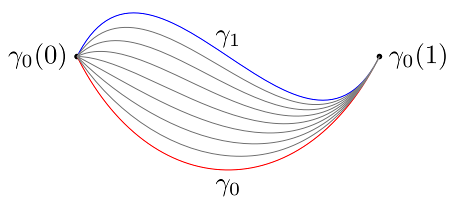
$$\begin{aligned} F(t,0) &= \gamma_0(t), \quad t \in [0,1] \\ F(t,1) &= \gamma_1(t), \quad t \in [0,1] \\ F(0,s) &= \gamma_0(0) = \gamma_1(0), \quad s \in [0,1] \\ F(1,s) &= \gamma_0(1) = \gamma_1(1), \quad s \in [0,1] \end{aligned}$$

In this case, say that F is a *(based) homotopy from* γ_0 *to* γ_1 and write $\gamma_0 \sim \gamma_1$.

The intuition here is that γ_0 may be continuously deformed into γ_1 through a collection of paths γ_s, $s \in [0,1]$, with the same end-points. Here γ_s is given by

$$\gamma_s(t) = F(t,s),$$

for $t \in [0,1]$. Here is a picture:



Before exploring this concept, we pause to record a technical lemma that we shall use about a zillion times.

Lemma 2.1. *Let X be a topological space and A, B closed subsets of X such that $A \cup B = X$. Let $\phi : X \to Y$ be a map into a topological space Y such that $\phi_{|A}$ and $\phi_{|B}$ are continuous with respect to the induced topologies on A and B respectively. Then ϕ is continuous on X.*

Proof. This is question 1 on exercise sheet 1. □

With this in hand, we justify our notation $\sim$ for the based homotopy relation:

Lemma 2.2. *The based homotopy relation $\sim$ is an equivalence relation.*

Proof.

Reflexive Let $\gamma : [0,1] \to X$ be any path, and define $F(t,s) = \gamma(t)$, $\forall (t,s) \in [0,1] \times [0,1]$. Then F is continuous being the composition of γ and $\pi_1 : [0,1] \times [0,1] \to [0,1]$. Moreover,

$$F(t,0) = \gamma(t), \quad F(t,1) = \gamma(t), \quad \forall t \in [0,1],$$

and

$$F(0,s) = \gamma(0), \quad F(1,s) = \gamma(1), \quad \forall s \in [0,1],$$

so that F is a based homotopy from γ to itself and $\gamma \sim \gamma$. Therefore, $\sim$ is reflexive.

Symmetric Assume that $\gamma_0, \gamma_1 : [0,1] \to X$ are paths such that $\gamma_0 \sim \gamma_1$ via a based homotopy F. Define a map $\tilde{F} : [0,1] \times [0,1] \to X$ by

$$\tilde{F}(t,s) = F(t, 1-s), \quad \forall (t,s) \in [0,1] \times [0,1].$$

Then $\tilde{F}$ is continuous, being the composition of F with $(t,s) \mapsto (t, 1-s)$. Moreover,

$$\tilde{F}(t,0) = F(t,1-0) = F(t,1) = \gamma_1(t), \quad \tilde{F}(t,1) = F(t,1-1) = F(t,0) = \gamma_0(t), \quad \forall t \in [0,1],$$

and

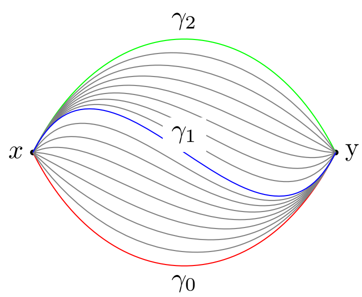
$$\tilde{F}(0,s) = F(0,1-s) = \gamma_0(0) = \gamma_1(0), \quad \forall s \in [0,1],$$

and

$$\tilde{F}(1,s) = F(1,1-s) = \gamma_0(1) = \gamma_1(1), \quad \forall s \in [0,1],$$

proving that $\gamma_1 \sim \gamma_0$ via the based homotopy $\tilde{F}$. We conclude that $\sim$ is symmetric.

Transitive Assume that $\gamma_0, \gamma_1, \gamma_2 : [0,1] \to X$ are paths in X such that $\gamma_0 \sim \gamma_1$ via a based homotopy F and $\gamma_1 \sim \gamma_2$ via G. The idea now is that we deform γ_0 into γ_2 by performing first F and then G:



Thus define a map $H : [0,1] \times [0,1] \to X$ by

$$H(t,s) = \begin{cases} F(t,2s), & (t,s) \in [0,1] \times [0,\frac{1}{2}] \\ G(t,2s-1), & (t,s) \in [0,1] \times [\frac{1}{2},1]. \end{cases}$$

Note that, despite having defined $H(t,s)$ twice when $s = \frac{1}{2}$, H is well-defined as the competing definitions coincide: $F(t,1) = \gamma_1(t) = G(t,0)$, for all $t \in [0,1]$. Moreover, observe that the restrictions of H to the closed subsets $[0,1] \times [0,\frac{1}{2}]$ and $[0,1] \times [\frac{1}{2},1]$ are continuous, being the composition of F or G with continuous maps. Thus Lemma 2.1 applies to show that H is continuous on $[0,1] \times [0,1]$.

It remains to check that H has the right values on the boundary of the square. For this,

$$H(t,0) = F(t,0) = \gamma_0(t), \quad \forall t \in [0,1], \quad H(t,1) = G(t,1) = \gamma_2(t), \quad \forall t \in [0,1],$$

and

$$H(0,s) = \begin{cases} F(0,2s), & s \in [0,\frac{1}{2}] \\ G(0,2s-1), & s \in [\frac{1}{2},1] \end{cases} = \gamma_0(0) = \gamma_2(0), \quad \forall s \in [0,1],$$

whereas

$$H(1,s) = \begin{cases} F(1,2s), & s \in [0,\frac{1}{2}] \\ G(1,2s-1), & s \in [\frac{1}{2},1] \end{cases} = \gamma_0(1) = \gamma_2(1), \quad \forall s \in [0,1].$$

Thus H is a based homotopy from γ_0 to γ_2 so that $\gamma_0 \sim \gamma_2$ and $\sim$ is transitive.

□

Definition. Let

$$\mathrm{Path}(X, x_0, y_0) = \{\gamma \in \mathcal{C}([0,1], X) \mid \gamma(0) = x_0, \gamma(1) = y_0\},$$

so that $\sim$ is an equivalence relation on $\mathrm{Path}(X, x_0, y_0)$. The set of equivalence classes (*based homotopy classes*) is denoted by $\pi_1(X, x_0, y_0)$.

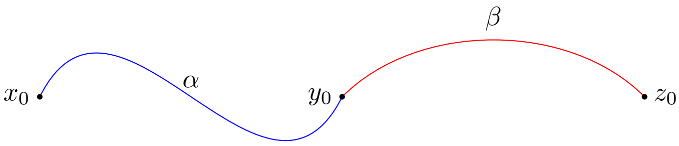
Our next major definition is that of a product of two paths.

Definition. Let $\alpha, \beta : [0,1] \to X$ be paths in X such that $\alpha(1) = \beta(0)$. The *product* or *join* $\alpha \cdot \beta : [0,1] \to X$ is defined by

$$(\alpha \cdot \beta)(t) = \begin{cases} \alpha(2t), & t \in [0, \frac{1}{2}] \\ \beta(2t-1), & t \in [\frac{1}{2}, 1]. \end{cases}$$

We remark that $\alpha \cdot \beta(t)$ is well-defined at $t = \frac{1}{2}$ since $\alpha(1) = \beta(0)$, and $\alpha \cdot \beta$ is continuous, thanks to Lemma 2.1, since its restrictions to $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$ are both continuous.

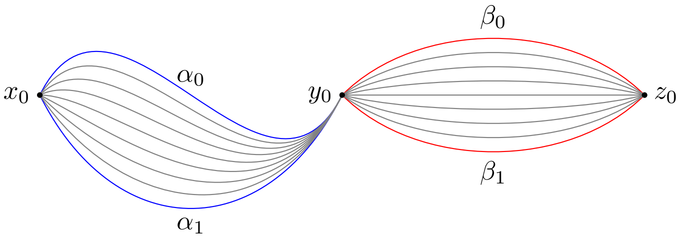
The intuition behind the definition is that $\alpha \cdot \beta$ is obtained by traversing α and then β. Both paths are traversed at twice the normal speed to ensure the the product is a map of the *unit* interval:



We now come to the main result of the section: our notions of based homotopy and joins of paths are compatible.

Theorem 2.3. *Let $\alpha_0, \alpha_1 : [0,1] \to X$ be paths from x_0 to y_0 in X and let $\beta_0, \beta_1 : [0,1] \to X$ be paths from y_0 to z_0 in X. If $\alpha_0 \sim \alpha_1$ and $\beta_0 \sim \beta_1$, then $\alpha_0 \cdot \beta_0 \sim \alpha_1 \cdot \beta_1$.*

Proof. Assume that $\alpha_0 \sim \alpha_1$ via a homotopy $F : [0,1] \times [0,1] \to X$ and that $\beta_0 \sim \beta_1$ via a homotopy $G : [0,1] \times [0,1] \to X$. We join F and G together in the same way we join paths:



So define $H : [0,1] \times [0,1] \to X$ by

$$H(t,s) = \begin{cases} F(2t, s), & (t,s) \in [0, \frac{1}{2}] \times [0,1] \\ G(2t-1, s), & (t,s) \in [\frac{1}{2}, 1] \times [0,1]. \end{cases}$$

We note that $H(t,s)$ is well-defined at $t = \frac{1}{2}$ since

$$F(2 \cdot \tfrac{1}{2}, s) = y_0 = G(2 \cdot \tfrac{1}{2} - 1, s).$$

Further, F and G are continuous maps of $[0,1] \times [0,1]$ so that the restrictions of H to the closed subsets $[0, \frac{1}{2}] \times [0,1]$ and $[\frac{1}{2}, 1] \times [0,1]$ are continuous also whence H is continuous on $[0,1] \times [0,1]$ by Lemma 2.1.

As for the values of H on the boundary of the unit square, we have,

$$H(t,0) = \begin{cases} F(2t,0), & t \in [0, \frac{1}{2}] \\ G(2t-1,0), & t \in [\frac{1}{2}, 1] \end{cases} = \begin{cases} \alpha_0(2t), & t \in [0, \frac{1}{2}] \\ \beta_0(2t-1), & t \in [\frac{1}{2}, 1] \end{cases} = (\alpha_0 \cdot \beta_0)(t), \quad \forall t \in [0,1].$$

Similarly,

$$H(t,1) = \begin{cases} F(2t,1), & t \in [0, \frac{1}{2}] \\ G(2t-1), & t \in [\frac{1}{2}, 1] \end{cases} = \begin{cases} \alpha_1(2t), & t \in [0, \frac{1}{2}] \\ \beta_1(2t-1), & t \in [\frac{1}{2}, 1] \end{cases} = (\alpha_1 \cdot \beta_1)(t), \quad \forall t \in [0,1].$$

Finally,

$$H(0,s) = F(0,s) = \alpha_0(0) = \alpha_1(0) = x_0 = (\alpha_0 \cdot \beta_0)(0) = (\alpha_1 \cdot \beta_1)(0),$$

and

$$H(1,s) = G(1,s) = \beta_0(1) = \beta_1(1) = z_0 = (\alpha_0 \cdot \beta_0)(1) = (\alpha_1 \cdot \beta_1)(1),$$

for all $s \in [0,1]$, proving that $\alpha_0 \cdot \beta_0 \sim \alpha_1 \cdot \beta_1$ via the homotopy H and we are done. □

The punchline is that we may define a multiplication of *based homotopy classes* of paths. For $[\alpha] \in \pi_1(X, x_0, y_0)$ and $[\beta] \in \pi_1(X, y_0, z_0)$, set

$$[\alpha] \cdot [\beta] := [\alpha \cdot \beta],$$

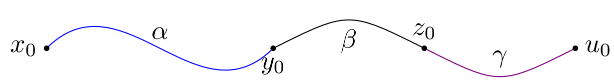
which, by Theorem 2.3, is well-defined (that is, independent of the choice of representatives α and β of the classes $[\alpha]$ and $[\beta]$).

We shall now show that this multiplication of homotopy classes is very well-behaved, in sharp contrast to that on paths.

2.2 The Fundamental Group

Notation. Henceforth, we denote the unit interval $[0,1]$ by I.

We begin by contemplating associativity of our multiplication. So let α, β, γ be paths whose product is defined:

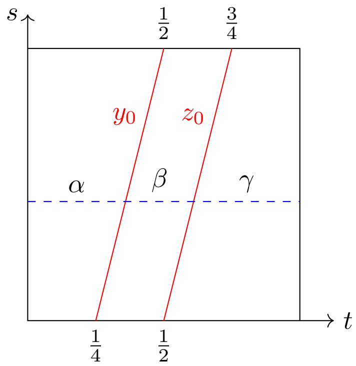


Observe that $(\alpha \cdot \beta) \cdot \gamma$ and $\alpha \cdot (\beta \cdot \gamma)$ are certainly not equal: their images are the same but their parametrisations are quite different. For example, the first traverses β when $t \in [\frac{1}{4}, \frac{1}{2}]$ and the second traverses β when $t \in [\frac{1}{2}, \frac{3}{4}]$.

However, the situation is much better when we consider *homotopy classes* of paths:

Lemma 2.4. *Let* $[\alpha] \in \pi_1(X, x_0, y_0)$, $[\beta] \in \pi_1(X, y_0, z_0)$ *and* $[\gamma] \in \pi_1(X, z_0, u_0)$. *Then*

$$([\alpha] \cdot [\beta]) \cdot [\gamma] = [\alpha] \cdot ([\beta] \cdot [\gamma]).$$



Proof. We must show that $(\alpha \cdot \beta) \cdot \gamma \sim \alpha \cdot (\beta \cdot \gamma)$. The strategy is to "stretch" the domains on which we traverse the three paths as shown in the diagram. Thus we define $F : I \times I \to X$ by

$$F(t,s) = \begin{cases} \alpha\left(\frac{4t}{1+s}\right), & 0 \le t \le \frac{s+1}{4} \\ \beta(4t-1-s), & \frac{s+1}{4} \le t \le \frac{s+2}{4} \\ \gamma\left(\frac{4t-(2+s)}{2-s}\right), & \frac{s+2}{4} \le t \le 1. \end{cases}$$

Note that F is well-defined: when $t = (s+1)/4$, both definitions yield $F(t,s) = y_0$ and, when $t = (s+2)/4$, both definitions give $F(t,s) = z_0$. Moreover, F is clearly continuous on the three closed sets into which we have divided $I \times I$ and so is continuous everywhere by Lemma 2.1. Finally, $F(0,s) = \alpha(0) = x_0$ and $F(1,s) = \gamma(1) = u_0$ while

$$F(t,0) = \begin{cases} \alpha(4t) & t \in [0, \frac{1}{4}] \\ \beta(4t-1) & t \in [\frac{1}{4}, \frac{1}{2}] \\ \gamma(2t-1) & t \in [\frac{1}{2}, 1] \end{cases} = (\alpha \cdot \beta) \cdot \gamma(t)$$

and, similarly, $F(t,1) = \alpha \cdot (\beta \cdot \gamma)$. Thus F is a based homotopy from $(\alpha \cdot \beta) \cdot \gamma$ to $\alpha \cdot (\beta \cdot \gamma)$. □

Our product has identity elements also.

Notation. For $x \in X$, let $\gamma_x : I \to X$ denote the constant path given by $\gamma_x(t) = x$. Moreover, let $1_x = [\gamma_x] \in \pi_1(X, x, x)$.

We now have:

Lemma 2.5. *Let $a \in \pi_1(X, x, y)$ and $b \in \pi_1(X, y, x)$. Then $1_x \cdot a = a$ and $b \cdot 1_x = b$.*

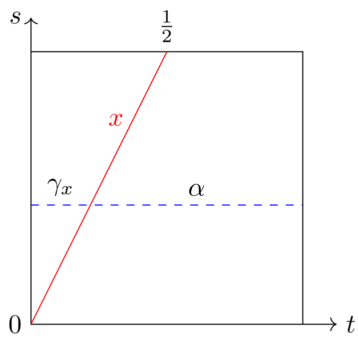
Proof. Let $a = [\alpha]$. We must prove that $\gamma_x \cdot \alpha \sim \alpha$. Our strategy will be to vary (linearly in s) the amount of time we spend at x before traversing α, as in the diagram on the right. So define $F : I \times I \to X$ by

$$F(t,s) = \begin{cases} x, & 0 \le t \le s/2 \\ \alpha\left(\frac{2t-s}{2-s}\right), & s/2 \le t \le 1. \end{cases}$$

Then F is well-defined as the competing definitions for $F(t,s)$ when $t = s/2$ both yield x. As usual, Lemma 2.1 shows that F is continuous. Finally, we easily check that

$$\begin{aligned} F(t,0) &= \alpha(t), \quad & F(t,1) &= \gamma_x \cdot \alpha(t), \\ F(0,s) &= x, & F(1,s) &= y, \end{aligned}$$

for all $t, s \in I$. Thus F is a based homotopy via which $\alpha \sim \gamma_x \cdot \alpha$. The argument for b is similar. □



It follows that 1_x behaves like the identity element of a group whenever multiplication with 1_x is defined. There are also inverses: given a path $\alpha : I \to X$, define $\bar{\alpha} : I \to X$ by $\bar{\alpha}(t) = \alpha(1-t)$, $\forall t \in I$.

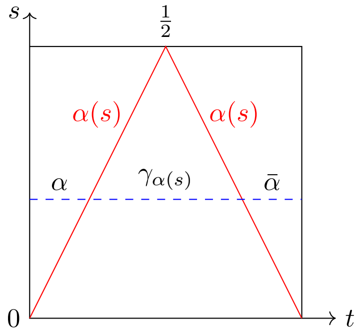
Exercise. Given paths $\alpha_0 \sim \alpha_1$, we have $\bar{\alpha}_0 \sim \bar{\alpha}_1$.

Thus $\alpha \mapsto \bar{\alpha}$ induces a well-defined map $a \mapsto \bar{a} : \pi_1(X, x, y) \to \pi_1(X, y, x)$.

We now have:

Lemma 2.6. *Let $a \in \pi_1(X, x, y)$ so that $\bar{a} \in \pi_1(X, y, x)$. Then $a \cdot \bar{a} = 1_x$ and $\bar{a} \cdot a = 1_y$.*

Proof. If $a = [\alpha]$, we must show that $\gamma_x \sim \alpha \cdot \bar{\alpha}$. The plan is to do α at twice the normal speed until reaching $\alpha(s)$, then hang around at $\alpha(s)$ until it is time to reverse our footsteps. So we divide up $I \times I$ as in the diagram on the right and define $F : I \times I \to X$ by



$$F(t,s) = \begin{cases} \alpha(2t), & 0 \leq t \leq s/2 \\ \alpha(s), & s/2 \leq t \leq 1 - s/2 \\ \alpha(2-2t), & 1 - s/2 \leq t \leq 1. \end{cases}$$

By now it should be clear what we have to do: check that F is well-defined ($F(t,s) = \alpha(s)$ where there are multiple definitions), use Lemma 2.1 to establish continuity of F and then check the values of F on the boundary of $I \times I$ to conclude that it is a based homotopy from γ_x to $\alpha \cdot \bar{\alpha}$.
Finally, replace a with $\bar{a}$, observing that $\bar{\bar{a}} = a$, to see that $\bar{a} \cdot a = 1_y$ also. □

Notation. In view of the above, we write a^{-1} instead of $\bar{a}$, for $a \in \pi_1(X, x, y)$.

In conclusion, our product satisfies most of the axioms of a group (actually, it yields the structure of a *groupoid*) except that the multiplication is not always defined. However, if we restrict attention to paths that begin and end at the same point x, the multiplication *is* always defined and we conclude:

Corollary 2.7. *The operation $\cdot$ endows the set of based homotopy classes $\pi_1(X, x, x)$ with the structure of a group.*

This triumph deserves celebrating with some terminology and notation:

Definition. A path $\gamma : I \to X$ is said to be a *loop* if $\gamma(0) = \gamma(1)$, and in this case γ is said to be *based at* $\gamma(0)$.

The set of all based homotopy classes of loops based at $x \in X$ is denoted by $\pi_1(X, x)$ (instead of $\pi_1(X, x, x)$) and is called *the fundamental group*[1] *of X based at x.*

At first sight, the fundamental group depends on the base point $x \in X$. However, for path-connected topological spaces, different base points yield isomorphic[2] groups.

Proposition 2.8. *Let $x, y \in X$, a path-connected topological space. Then $\pi_1(X, x) \cong \pi_1(X, y)$.*

Proof. The basic observation here is that if α is a loop based at x and γ is a path from x to y, then $\gamma^{-1} \cdot \alpha \cdot \gamma$ is a loop at y (draw a picture!). This will give us our isomorphism $\pi_1(X, x) \to \pi(X, y)$.

Since X is path-connected, $\pi_1(X, x, y)$ is non-empty so choose $g \in \pi_1(X, x, y)$ and define $g_* : \pi_1(X, x) \to \pi_1(X, y)$ by

$$g_*(a) = g^{-1} \cdot a \cdot g \in \pi_1(X, y),$$

for $a \in \pi_1(X)$. Then g_* is a homomorphism:

$$g_*(a \cdot b) = g^{-1} \cdot (a \cdot b) \cdot g = (g^{-1} \cdot a \cdot g) \cdot (g^{-1} \cdot b \cdot g) = g_*(a) \cdot g_*(b),$$

for $a, b \in \pi_1(X)$. Moreover, note that

$$((g^{-1})_* \circ g_*)(a) = g \cdot g^{-1} \cdot a \cdot g \cdot g^{-1} = a,$$

for $a \in \pi_1(X, x)$, so that $(g^{-1})_* \circ g_* = \mathrm{id}_{\pi_1(X,x)}$ and similarly $g_* \circ (g^{-1})_* = \mathrm{id}_{\pi_1(X,y)}$. Therefore, g_* is an isomorphism with inverse $(g^{-1})_*$ (that is, $(g_*)^{-1} = (g^{-1})_*$). □

[1] $\pi_1(X, x)$ is also sometimes called the *Poincaré group* of X after its inventor Henri Poincaré (1854–1912).

[2] Recall: a map $\phi : G_1 \to G_2$ of groups is a *homomorphism* if it preserves the group laws: $\phi(gh) = \phi(g)\phi(h)$, for $g, h \in G_1$. It is an *isomorphism* if it is a bijective homomorphism. In this case, ϕ^{-1} is also an isomorphism and we say that G_1 and G_2 are *isomorphic* and write $G_1 \cong G_2$.

Remarks.

1. If $x, y \in X$ lie in different path-components, then $\pi_1(X,x) \ncong \pi_1(X,y)$, in general. For example, take $X = S^1 \cup \{0\} \subset \mathbb{C}$ and $x = 1$, $y = 0$.

2. If we choose a different $g' \in \pi_1(X,x,y)$, then, in general, $g_* \neq g'_*$.

2.3 Continuous Maps and the Fundamental Group

Let us now consider how continuous maps $\phi : X \to Y$ of topological spaces relate the fundamental groups of X and Y.

Proposition 2.9. *Let $\gamma_0, \gamma_1 : I \to X$ be paths in X and $\phi : X \to Y$ continuous.*

(i) If $\gamma_0 \sim \gamma_1$, then $\phi \circ \gamma_0 \sim \phi \circ \gamma_1$.

(ii) If $\gamma_0(1) = \gamma_1(0)$, then $\phi \circ (\gamma_0 \cdot \gamma_1) = (\phi \circ \gamma_0) \cdot (\phi \circ \gamma_1)$.

(iii) $\phi \circ \overline{\gamma}_0 = \overline{(\phi \circ \gamma_0)}$.

(iv) $\phi \circ \gamma_x = \gamma_{\phi(x)}$.

Proof. Easy exercises! □

As a result of part (i), $\phi : X \to Y$ induces a well-defined map $\phi_* : \pi_1(X,x) \to \pi_1(Y,\phi(x))$ by

$$\phi_*([\gamma]) = [\phi \circ \gamma],$$

for $\gamma : I \to X$ a loop based at $x \in X$. Then, part (ii) tells us:

Corollary 2.10. *$\phi_* : \pi_1(X,x) \to \pi_1(Y,\phi(x))$ is a homomorphism of groups.*

In fact, more is true: if $\psi : Y \to Z$ is also continuous, then, for all $[\gamma] \in \pi_1(X,x)$,

$$(\psi \circ \phi)_*([\gamma]) = [(\psi \circ \phi) \circ \gamma] = [\psi \circ (\phi \circ \gamma)] = \psi_*([\phi \circ \gamma]) = \psi_*(\phi_*([\gamma])),$$

proving that $(\psi \circ \phi)_* = \psi_* \circ \phi_*$. Moreover,

$$(\mathrm{id}_X)_*[\gamma] = [\mathrm{id}_X \circ \gamma] = [\gamma],$$

so that $(\mathrm{id}_X)_* = \mathrm{id}_{\pi_1(X,x)}$. Therefore, the induced "mapping" $\phi \mapsto \phi_*$ from continuous maps to group homomorphisms preserves composition and identities. In the language of category theory, such a map is called a *functor*.

Remark. Be warned: set-theoretic properties of maps need not behave well under $\phi \mapsto \phi_*$: if ϕ injects/surjects/bijects, it does not follow that ϕ_* has the same property.

As a corollary to the above development, we have:

Theorem 2.11. *If $\phi : X \to Y$ is a homeomorphism, then for all $x \in X$, the map $\phi_* : \pi_1(X,x) \to \pi_1(Y,\phi(x))$ is an isomorphism.*

Proof. Since $\phi : X \to Y$ is a homeomorphism, we have

$$\phi^{-1} \circ \phi = \mathrm{id}_X, \quad \phi \circ \phi^{-1} = \mathrm{id}_Y.$$

Applying $*$ to both sides gives

$$(\phi^{-1})_* \circ \phi_* = \mathrm{id}_{\pi_1(X,x)}, \quad \phi_* \circ (\phi^{-1})_* = \mathrm{id}_{\pi_1(Y,\phi(x))},$$

so that $\phi_* : \pi_1(X,x) \to \pi_1(Y,\phi(x))$ is a bijection with inverse $(\phi_*)^{-1} = (\phi^{-1})_*$. Thus ϕ_* is an isomorphism. □

The punchline here is that the fundamental group is a *topological invariant*: homeomorphic spaces have isomorphic fundamental groups. However, as we shall see, quite different looking spaces can have the same fundamental group. This is ultimately because, just as we can deform loops, we can also deform continuous maps.

Definition. Two continuous maps $\phi_0, \phi_1 : X \to Y$ of topological spaces are said to be *homotopic* if there exists a continuous map $F : X \times I \to Y$ (here $X \times I$ has the product topology) such that

$$F(x, 0) = \phi_0(x), \quad F(x, 1) = \phi_1(x), \quad \forall x \in X.$$

In this case, we write: $\phi_0 \simeq \phi_1$ and say that F is a *homotopy from ϕ_0 to ϕ_1*.

The intuition here is that $\phi_0 : X \to Y$ may be continuously deformed into $\phi_1 : X \to Y$ through continuous maps $\phi_s : X \to Y$, $s \in [0, 1]$, where $\phi_s(x) = F(x, s)$, $\forall x \in X, s \in I$.

Exercise. $\simeq$ is an equivalence relation on $\mathcal{C}(X, Y)$.

The equivalence classes are called *homotopy classes* of continuous maps.

The notion of homotopy just given is slightly different from the previous notion of *based* homotopy: there is no analogue of the requirement that end-points be preserved. For this, we have:

Definition. Let $A \subset X$ and $\phi_0, \phi_1 : X \to Y$ be continuous maps. Then, ϕ_0 is said to be *homotopic to ϕ_1 relative to A* if there exists a continuous map $F : X \times I \to Y$ such that

$$\begin{aligned} F(x, 0) &= \phi_0(x), \\ F(x, 1) &= \phi_1(x), \\ F(a, s) &= \phi_0(a) = \phi_1(a), \end{aligned}$$

for all $x \in X$, $a \in A$, $s \in I$. (In particular, $\phi_{0|A} = \phi_{1|A}$.)

In this case, we write $\phi_0 \simeq \phi_1$ rel A and say that F is a *homotopy relative to A from ϕ_0 to ϕ_1*.

Example. Two paths $\gamma_0, \gamma_1 : I \to X$ satisfy $\gamma_0 \sim \gamma_1$ if and only if $\gamma_0 \simeq \gamma_1$ rel $\{0, 1\}$.

Theorem 2.12. *If $\phi_0, \phi_1 : X \to Y$ are homotopic relative to $\{x\}$, for some $x \in X$, then*

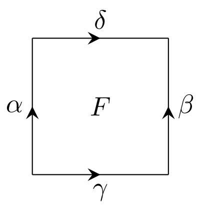
$$(\phi_0)_* = (\phi_1)_* : \pi_1(X, x) \to \pi_1(Y, \phi_0(x)) = \pi_1(Y, \phi_1(x)).$$

Proof. Let γ be a loop based at $x \in X$. We must show that $\phi_0 \circ \gamma \sim \phi_1 \circ \gamma$. However, if $F : X \times I \to Y$ is a homotopy between ϕ_0 and ϕ_1 relative to $\{x\}$, then the map $G : I \times I \to Y$ defined by

$$G(t, s) = F(\gamma(t), s),$$

for $t, s \in I$, is easily checked to be a based homotopy from $\phi_0 \circ \gamma$ to $\phi_1 \circ \gamma$. □

It would be better to drop the "relative to $\{x\}$" restriction in Theorem 2.12 and this can be done. However, a little care is required since, in general, the loops $\phi_0 \circ \gamma$, $\phi_1 \circ \gamma$ and all the loops interpolating between them could have different base points. So we start with a lemma:



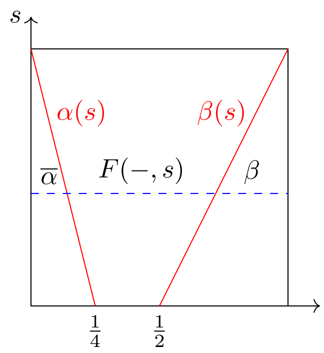
Lemma 2.13. *Let $F : I \times I \to X$ be continuous. Define paths $\alpha, \beta, \gamma, \delta : I \to X$ as in the diagram on the left. Thus*

$$\alpha(s) = F(0, s), \quad \beta(s) = F(1, s), \quad \gamma(t) = F(t, 0), \quad \delta(t) = F(t, 1),$$

for $t, s \in I$.
Then $\delta \sim \overline{\alpha} \cdot \gamma \cdot \beta$.

Proof. We prove that $(\bar{\alpha}\cdot\gamma)\cdot\beta \sim \delta$. The plan is shown in the picture on the right: traverse $\bar{\alpha}$ reaching $\alpha(s)$ when $t = (1-s)/4$, then do $F(-,s)$ to reach $\beta(s)$ at $t = (1+s)/2$ and finally do β the rest of the way. So we define $H : I \times I \to X$ by

$$H(t,s) = \begin{cases} \bar{\alpha}(4t), & 0 \le t \le \frac{1-s}{4} \\ F\big(\frac{4t+s-1}{3s+1}, s\big), & \frac{1-s}{4} \le t \le \frac{1+s}{2} \\ \beta(2t-1), & \frac{1+s}{2} \le t \le 1. \end{cases}$$



As usual, there is lots to check and it is a straightforward exercise to do that checking to conclude that H is a based homotopy from $(\bar{\alpha}\cdot\gamma)\cdot\beta$ to δ. □

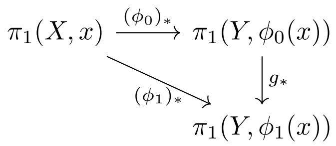
We put this to work to examine the effect of homotopic maps on $\pi_1(X,x)$. First recall from Proposition 2.8 that a class of paths $g \in \pi_1(X,x,y)$ induces an isomorphism

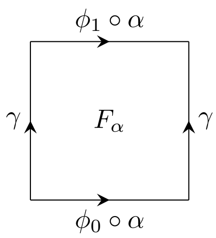
$$g_* : \pi_1(X,x) \cong \pi_1(X,y)$$

defined by $a \mapsto g^{-1}\cdot a\cdot g$, for $a \in \pi_1(X,x)$. With this in mind, we have:

Theorem 2.14. *Let $\phi_0, \phi_1 : X \to Y$ be homotopic via a homotopy $F : X \times I \to Y$ and let $x \in X$. Let $\gamma : I \to Y$ be the path in Y from $\phi_0(x)$ to $\phi_1(x)$ given by $\gamma(s) = F(x,s)$, for $s \in I$, and set $g = [\gamma]$.*

Then $(\phi_1)_* = g_* \circ (\phi_0)_*$ so that we have a commuting diagram:





Proof. Let $\alpha : I \to X$ be a loop based at $x \in X$ and contemplate the map $F_\alpha : I \times I \to Y$ defined by $F_\alpha(t,s) = F(\alpha(t),s)$. The values of F_α on the boundary of $I \times I$ are shown on the left. Thus Lemma 2.13 applies to show that $\phi_1 \circ \alpha \sim \bar{\gamma}\cdot(\phi_0\circ\alpha)\cdot\gamma$, so that

$$(\phi_1)_*[\alpha] = (g_* \circ (\phi_0)_*)[\alpha]$$

□

Thus $(\phi_0)_*$ and $(\phi_1)_*$ differ by an isomorphism. In particular:

Corollary 2.15. *$(\phi_0)_* : \pi_1(X,x) \to \pi_1(Y,\phi_0(x))$ is an isomorphism if and only if $(\phi_1)_* : \pi_1(X,x) \to \pi_1(Y,\phi_1(x))$ is an isomorphism.*

We have seen that homeomorphisms induce isomorphisms on π_1 but the above development indicates that this should also be true for continuous maps that are only invertible *up to homotopy*. Here is the definition that captures this idea:

Definition. Two topological spaces X and Y are said to be *homotopy equivalent* (or *have the same homotopy type*) if there exist continuous maps $\phi : X \to Y$ and $\psi : Y \to X$ such that $\psi\circ\phi \simeq \mathrm{id}_X$ and $\phi\circ\psi \simeq \mathrm{id}_Y$.

In this case, we write $X \simeq Y$ and say that ϕ is a *homotopy equivalence* with *homotopy inverse* ψ.

Theorem 2.16. *Let $\phi : X \to Y$ be a homotopy equivalence. Then $\phi_* : \pi_1(X,x) \to \pi_1(Y,\phi(x))$ is an isomorphism for any $x \in X$.*

Thus homotopy-equivalent path-connected spaces have isomorphic fundamental groups.

Proof. Let ψ be a homotopy inverse of ϕ. Then Corollary 2.15 tells us that $\psi_*\circ\phi_* = (\psi\circ\phi)_*$ is an isomorphism since $(\mathrm{id}_X)_*$ is. Thus ϕ_* injects. Similarly, $\phi_*\circ\psi_*$ is an isomorphism so ϕ_* surjects. □

Examples.

1. Clearly, if $X \cong Y$ then $X \simeq Y$.
2. The converse is not true: the open ball $B^n \subset \mathbb{R}^n$ is homotopy equivalent to a singleton set! Indeed, let $X = \{0\}$, take $\phi : X \to B^n$ to be the inclusion map, and define $\psi : B^n \to X$ to be the only thing it can be: $\psi : x \mapsto 0$. Then $\psi \circ \phi = \mathrm{id}_X$, while $\phi \circ \psi : B^n \to B^n$ is constant. However, $\phi \circ \psi \simeq \mathrm{id}_{B^n}$ via the homotopy $F(x, s) = sx$.

This last is a special case of the following setup:

Definition. Let X be a topological space and let $A \subset X$ with inclusion $i : A \to X$. Say that A is a *retract* of X if there exists a continuous map $r : X \to A$ such that $r(a) = a$, $\forall a \in A$, that is, $r \circ i = \mathrm{id}_A$. In this case, the map $r : X \to A$ is called a *retraction*.

A is said to be a *deformation retract* of X if there exists a retraction $r : X \to A$ such that $i \circ r \simeq \mathrm{id}_X$ rel A.

Thus, A is a deformation retract of X if there exists a continuous map $F : X \times I \to X$ such that

$$F(x, 0) = x, \quad F(x, 1) = r(x), \quad F(a, s) = a,$$

for all $x \in X$, $a \in A$ and $s \in I$.

It follows at once that the retraction of a deformation retract is a homotopy equivalence with homotopy inverse i. In particular, $\pi_1(A, a) \cong \pi_1(X, a)$ by Theorem 2.16.

Example. We have just seen that $\{0\}$ is a deformation retract of B^n.

Exercises.

1. Let $X \subseteq \mathbb{R}^n$ be convex, and let $x \in X$. Then $\{x\}$ is a deformation retract of X.
   Thus $\pi(X, x) \cong \pi_1(\{x\}, x) \cong \{1\}$, the trivial group.
2. The $(n-1)$-dimensional sphere S^{n-1} is a deformation retract of $\mathbb{R}^n \setminus \{0\}$.

This circle of ideas deserves some terminology:

Definition. A topological space X is said to be *simply connected* if it is path-connected and $\pi_1(X) = \{1\}$.

X is said to be *contractible* if X is homotopy equivalent to a singleton set: $X \simeq \{\mathrm{pt}\}$.

Exercise. Contractible spaces are simply connected. (The issue here is to see that contractible spaces are path-connected.)

2.4 Calculating the Fundamental Group

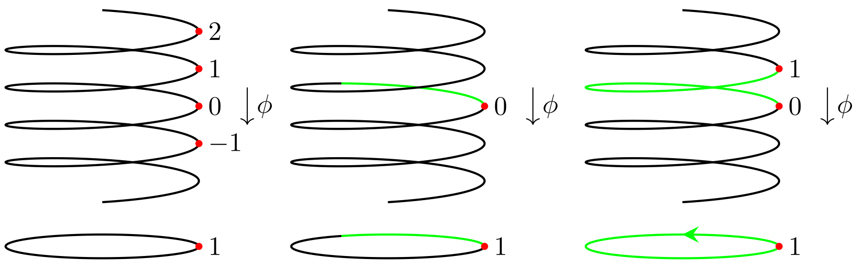
So far, the only fundamental groups we can compute (those of convex subsets of $\mathbb{R}^n$) are trivial. We now turn to some techniques for calculating the fundamental group. The most substantial example is that of the circle $S^1 \subset \mathbb{C}$.

Theorem 2.17. *$\pi_1(S^1, 1) \cong \mathbb{Z}$.*

The intuition here is that the homotopy class of a loop in S^1 depends only on how many times it wraps around the circle and in which direction. To make sense of "the number of times a loop wraps around the circle" is the first challenge we must overcome. For this, we introduce the map $\phi : \mathbb{R} \to S^1$ defined by

$$\phi(t) = e^{2\pi i t},$$

for $t \in \mathbb{R}$. One can visualise ϕ as projection from a helical embedding of $\mathbb{R}$ as in Figure 2.1a.



(a) The map ϕ (b) Tracking a path (c) Tracking a loop

Figure 2.1: Covering S^1

Now contemplate a path σ in S^1, starting at 1. It seems plausible (and we shall prove it in Lemma 2.18 below) that there should be a path σ' in $\mathbb{R}$ starting at 0 that tracks the progress of σ in the sense that $\phi \circ \sigma' = \sigma$ (see Figure 2.1b). If σ is a loop, something interesting happens: the tracking path must end at an integer: $\phi(\sigma'(1)) = 1$ so that $\sigma' \in \phi^{-1}\{1\} = \mathbb{Z}$, and experimentation shows that this integer coincides with the number of times σ has wrapped around the circle anti-clockwise as in Figure 2.1c (draw some pictures of your own!). This gives us a practical handle on this wrapping number and is the key to proving Theorem 2.17.

With all this in mind, let us collect the main properties of ϕ that we shall exploit:

1. $\phi : \mathbb{R} \to S^1$ is continuous;
2. $\phi : \mathbb{R} \to S^1$ is a group homomorphism, since

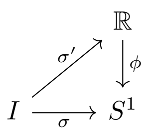
$$\phi(t + s) = e^{2\pi i(t+s)} = e^{2\pi it}e^{2\pi is} = \phi(t)\phi(s),$$

   for $t, s \in \mathbb{R}$;
3. $\phi(t) = 1$ if and only if $t \in \mathbb{Z}$, otherwise said, $\operatorname{Ker} \phi = \mathbb{Z}$;
4. $\phi : \mathbb{R} \to S^1$ is an open map (exercise!);
5. (the most important bit) $\phi|_{(-\frac{1}{2},\frac{1}{2})} : (-\frac{1}{2}, \frac{1}{2}) \to S^1 \setminus \{-1\}$ is a bijection and so a homeomorphism.

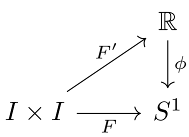
   We let $\psi : S^1 \setminus \{-1\} \to (-\frac{1}{2}, \frac{1}{2})$ be the inverse map.

We need two lemmata:

Lemma 2.18 (Path-Lifting Lemma). *If $\sigma : I \to S^1$ is a path beginning at $1 \in S^1$, then there exists a* unique *path $\sigma' : I \to \mathbb{R}$ starting at 0 such that $\phi \circ \sigma' = \sigma$. Thus we have a commuting diagram:*



Lemma 2.19 (Homotopy-Lifting Lemma). *If $\tau : I \to S^1$ is another path beginning at 1, and $\sigma \sim \tau$ via a homotopy $F : I \times I \to S^1$, then $\sigma' \sim \tau'$ via a unique homotopy $F' : I \times I \to \mathbb{R}$ such that $\phi \circ F' = F$. Thus we have a commuting diagram:*



Proof. We will prove both lemmata in one shot. For this, let Y be either I or $I \times I$ and, accordingly, take $0 \in Y$ to be either $0 \in I$ or $(0,0) \in I \times I$. Now let $f : Y \to S^1$ be either σ or F. We will construct a continuous $f' : Y \to \mathbb{R}$ such that $\phi \circ f' = f$ and $f'(0) = 0$.

To do this, we begin by noting that Y is a compact metric space so that f is *uniformly* continuous. This means that there is $\delta > 0$ such that, for all $y, y' \in Y$, whenever $|y - y'| < \delta$, we have $|f(y) - f(y')| < 1$ and, in particular, $f(y) \neq -f(y')$ so that $f(y)/f(y') \in S^1 \setminus \{-1\}$ and $\psi(f(y)/f(y')) \in \mathbb{R}$ is defined.

Now fix $N \in \mathbb{N}$ large enough that $|y| < N\delta$, for all $y \in Y$, and define $f' : Y \to \mathbb{R}$ by

$$f'(y) = \psi\big(f(y)/f(\tfrac{N-1}{N}y)\big) + \psi\big(f(\tfrac{N-1}{N}y)/f(\tfrac{N-2}{N}y)\big) + \cdots + \psi\big(f(\tfrac{1}{N}y)/f(0)\big),$$

for $y \in Y$.

Observe:

1. Each summand $y \mapsto \psi\big(f(\frac{k}{N}y)/f(\frac{k-1}{N}y)\big)$, $1 \leq k \leq N$, is well-defined, since $|\frac{k}{N}y - \frac{k-1}{N}y| = |y|/N < \delta$, and continuous. Thus $f' : Y \to \mathbb{R}$ is a continuous function.
2. $f'(0) = \psi\big(f(0)/f(0)\big) + \cdots + \psi\big(f(0)/f(0)\big) = N\psi(1) = 0$.
3. For any $y \in Y$, we have

$$\begin{aligned}(\phi \circ f')(y) &= \prod_{k=1}^{N} \phi\Big(\psi\big(f(\tfrac{k}{N}y)/f(\tfrac{k-1}{N}y)\big)\Big), && \text{since } \phi \text{ is a homomorphism}\\ &= \prod_{k=1}^{N} f(\tfrac{k}{N}y)/f(\tfrac{k-1}{N}y), && \text{since } \phi \circ \psi = \mathrm{id}\\ &= f(y)/f(0) = f(y),\end{aligned}$$

   since $f(0) = 1$.

Moreover, $f' : Y \to \mathbb{R}$ is the unique map with these properties. Indeed, if $f'' : Y \to \mathbb{R}$ is a continuous map with $f''(0) = 0$ and $\phi \circ f'' = f$, then, since ϕ is a homomorphism,

$$\phi((f' - f'')(y)) = \frac{(\phi \circ f')(y)}{(\phi \circ f'')(y)} = 1,$$

for $y \in Y$, so that $\operatorname{Im}(f' - f'') \subset \mathbb{Z}$ because $\operatorname{Ker}\phi = \mathbb{Z}$. But any continuous map of a connected space Y into $\mathbb{Z}$ must be constant, so that $f' - f''$ is constant. However, $f'(0) = f''(0) = 0$, so $f' = f''$.

Now take $Y = I$, $f = \sigma$ and then $\sigma' = f'$ to settle 2.18.

For the homotopy-lifting lemma, take $Y = I \times I$, $f = F$ and set $F' = f'$. We must show that F' is a based homotopy between σ' and τ' and so must examine the values of F on the boundary of $I \times I$. So define paths by

$$\alpha(s) = F'(0,s), \quad \beta(s) = F'(1,s), \quad \gamma(t) = F'(t,0), \quad \delta(t) = F'(t,1),$$

for $s, t \in I$. We need to show that $\delta = \tau'$, $\gamma = \sigma'$, $\alpha \equiv 0$ and β is constant. Firstly, $\gamma(0) = F'(0,0) = 0$ while, for all $t \in I$, $\phi \circ \gamma(t) = F(t,0) = \sigma(t)$ so that $\phi \circ \gamma = \sigma$. The uniqueness assertion in 2.18 now tells us that $\gamma = \sigma'$.

Further, $\alpha(0) = 0$ while $\phi \circ \alpha(s) = F(0,s) = 1$, for all $s \in I$, that is, $\phi \circ \alpha = \gamma_1$, the constant path at $1 \in S^1$. On the other hand, the constant path $\gamma_0 : I \to \mathbb{R}$ also has these properties so that the uniqueness part of 2.18 now gives $\alpha = \gamma_0$, that is, $\alpha \equiv 0$.

Now $\delta(0) = \alpha(1) = 0$ and $\phi \circ \delta = \tau$ so the same argument gives us that $\delta = \tau'$.

Finally, consider $\hat{\beta} := \beta - \beta(0)$. We have $\hat{\beta}(0) = 0$ and

$$\phi \circ \hat{\beta} = (\phi \circ \beta)/(\phi \circ \beta(0)) = \sigma(1)/\sigma(1) = 1.$$

However, the constant path γ_0 also has these properties so a final appeal to the uniqueness part of 2.18 yields $\beta - \beta(0) \equiv 0$ and we are done. □

As a corollary, we see that since $\sigma' \sim \tau'$, we must have $\sigma'(1) = \tau'(1)$:

Corollary 2.20. *$\sigma'(1)$ depends only on the based homotopy class of σ.*

After all this, we can finally prove Theorem 2.17:

Proof of Theorem 2.17. Let σ be a loop in S^1 based at 1. Then $\sigma(1) = 1$ so that $\sigma'(1) \in \mathbb{Z}$. We therefore define a map $\chi : \pi_1(S^1, 1) \to \mathbb{Z}$ by

$$\chi([\sigma]) = \sigma'(1).$$

χ is well defined by Corollary 2.20 and I claim that χ is an isomorphism of groups.

First we show that it is a homomorphism. So let $[\sigma], [\tau] \in \pi_1(S^1, 1)$, and write $\sigma'(1) = m$ and $\tau'(1) = n$. Consider the path $\tau'' : I \to \mathbb{R}$ defined by $\tau''(t) = \tau'(t) + m$. We have:

(a) $\phi \circ \tau'' = \phi \circ \tau' = \tau$, since $\phi(m) = 1$ and ϕ is a homomorphism;

(b) $\tau''(0) = m = \sigma'(1)$, so the product $\sigma' \cdot \tau''$ is defined, and

$$\phi \circ (\sigma' \cdot \tau'') = (\phi \circ \sigma') \cdot (\phi \circ \tau'') = \sigma \cdot \tau;$$

(c) $(\sigma' \cdot \tau'')(0) = \sigma'(0) = 0$.

Now the uniqueness part of 2.18 kicks in to tell us that $(\sigma \cdot \tau)' = \sigma' \cdot \tau''$. Thus,

$$\chi([\sigma][\tau]) = \chi([\sigma \cdot \tau]) = (\sigma \cdot \tau)'(1) = (\sigma' \cdot \tau'')(1) = \tau''(1) = \tau'(1) + m = n + m = \chi([\sigma]) + \chi([\tau])$$

and χ is a homomorphism.

To see that χ surjects, pick $n \in \mathbb{Z}$ and let $\sigma : I \to S^1$ be given by $\sigma(t) = \phi(nt)$, $t \in I$. Then σ is a loop based at 1 and $\sigma'(t) = nt$ (why?) so that $\chi([\sigma]) = n$.

Finally, we show χ injects. Since χ is a homomorphism, it suffices to show that $\ker \chi$ is trivial. But if $\chi[\sigma] = 0$, this means that $\sigma'(1) = 0$ so that σ' is a loop in $\mathbb{R}$ based at zero. But $\mathbb{R}$ is contractible and so simply connected whence $\sigma' \sim \gamma_0$. Thus $\sigma = \phi \circ \sigma' \sim \phi \circ \gamma_0 = \gamma_1$. Otherwise said, $[\sigma] = 1_1 \in \pi(S^1, 1)$ and we are done.

□

Other Calculations

In contrast to S^1, the higher-dimensional spheres S^n (for $n \geq 2$) are simply connected.

To prove this, we need a technical lemma:

Lemma 2.21 (Lebesgue Covering Lemma)**.** *Let (X, d) be a compact metric space and let $\{U_\alpha\}_{\alpha \in I}$ be an open cover of X. Then $\exists \delta > 0$ such that any subset of diameter strictly less than δ is contained in one of the sets U_α.*

Such a δ is said to be a Lebesgue number *of the cover.*

Proof. Exercise! □

Theorem 2.22. *Let X be a topological space, and suppose that $X = U \cup V$ where $U, V \subset X$ are open and simply connected, while $U \cap V$ is non-empty and path-connected. Then X is simply connected.*

Proof. Choose a base point $x_0 \in U \cap V$. Since U, V are path-connected and $X = U \cup V$, we can find a path from any $x \in X$ to x_0 so that the path component of x_0 is all of X.

Our main task is therefore to show that $\pi_1(X, x_0) \cong \{1\}$. So let α be a loop based at x_0. Our plan is to write α as a product

$$\alpha \sim \beta_1 \cdot \dots \cdot \beta_N,$$

where each β_k is a loop in U or a loop in V based at x_0. Then each $\beta_k \sim \gamma_{x_0}$ since U, V are simply connected, whence $\alpha \sim \gamma_{x_0}$ and $[\alpha] = 1_{x_0}$.

Now $\{\alpha^{-1}(U), \alpha^{-1}(V)\}$ is an open cover of I which is a compact metric space. Thus, the Lebesgue Covering Lemma 2.21 ensures the existence of a partition

$$0 = s_0 < s_1 < \dots < s_N = 1$$

with each $\alpha([s_{k-1}, s_k]) \subset U$ or $\alpha([s_{k-1}, s_k]) \subset V$ (choose the mesh size of the partition to be less than the Lebesgue number of the cover[3]).

Define $\alpha_k : I \to X$, $1 \leq k \leq N$, by

$$\alpha_k(t) = \alpha(ts_k + (1-t)s_{k-1}),$$

for $t \in I$. Then $\alpha \sim \alpha_1 \cdot \dots \cdot \alpha_N$ and each α_k has image in U or image in V. Now let γ_k be a path from x_0 to $\alpha(s_k)$ with image in U, V or $U \cap V$ according to whether $\alpha(s_k) \in U, V$ or $U \cap V$ (this is possible since U, V and $U \cap V$ are all path-connected), and take $\gamma_0 = \gamma_N = \gamma_{x_0}$. Now,

$$\alpha \sim (\gamma_0 \cdot \alpha_1 \cdot \gamma_1^{-1}) \cdot (\gamma_1 \cdot \alpha_2 \cdot \gamma_2^{-1}) \cdot \dots \cdot (\gamma_{N-1} \cdot \alpha_N \cdot \gamma_N^{-1}),$$

and each path $\beta_k := \gamma_{k-1} \cdot \alpha_k \cdot \gamma_k^{-1}$ is a loop at x_0 with image in U or V. Thus

$$\alpha \sim \beta_1 \cdot \dots \cdot \beta_N,$$

and we are done. □

Corollary 2.23. *The n-sphere S^n is simply connected, for $n \geq 2$.*

Proof. The key point here is that, for any $x \in S^n$, $S^n \setminus \{x\} \cong \mathbb{R}^n$ via stereoprojection: see Figure 2.2. So let $x \neq y$ be distinct elements of S^n and set $U = S^n \setminus \{x\}$, $V = S^n \setminus \{y\}$. Then $U, V \cong \mathbb{R}^n$ and so

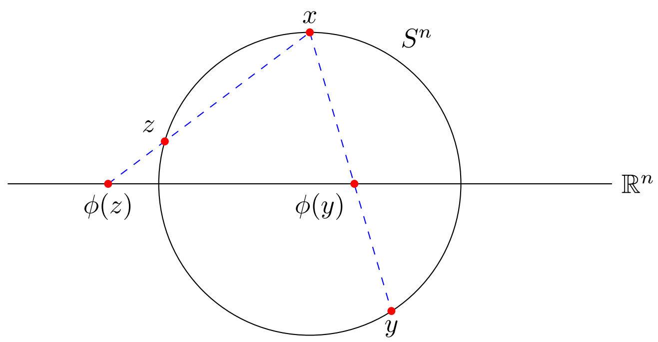


Figure 2.2: $S^n \setminus \{x\} \cong \mathbb{R}^n$ via stereoprojection

are simply connected while $U \cap V \cong \mathbb{R}^n \setminus \{\text{pt}\}$ which is non-empty and path-connected (this is where we use $n \geq 2$). Thus 2.22 applies and we are done. □

Theorem 2.24. *Let X and Y be topological spaces and, as usual, equip $X \times Y$ with the product topology. Then*

$$\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0),$$

for any $(x_0, y_0) \in X \times Y$.

[3] In more detail, with $\delta > 0$ a Lebesgue number of the cover, choose the partition so that $\max_{1 \leq k \leq N} |s_k - s_{k-1}| < \delta$.

Recall that the group multiplication on a Cartesian product $G_1 \times G_2$ of groups is given component-wise:

$$(g_1, g_2)(h_1, h_2) = (g_1 h_1, g_2 h_2),$$

for $g_i, h_i \in G_i$.

Proof. The continuous projections $p_1 : X \times Y \to X$, $p_2 : X \times Y \to Y$ induce homomorphisms

$$(p_1)_* : \pi_1(X \times Y, (x_0, y_0)) \to \pi_1(X, x_0), \quad (p_2)_* : \pi_1(X \times Y, (x_0, y_0)) \to \pi_1(Y, y_0),$$

and so we get a homomorphism $\phi : \pi_1(X \times Y, (x_0, y_0)) \to \pi_1(X, x_0) \times \pi_1(Y, y_0)$ given by

$$\phi([\alpha]) = ((p_1)_*([\alpha]), (p_2)_*([\alpha])),$$

for all $[\alpha] \in \pi_1(X \times Y, (x_0, y_0))$.

ϕ **surjects**: let $([\beta], [\gamma]) \in \pi_1(X, x_0) \times \pi_1(Y, y_0)$; then $\alpha = (\beta, \gamma)$ is a loop in $X \times Y$ based at (x_0, y_0), and $\phi([\alpha]) = ([\beta], [\gamma])$.

ϕ **injects**: ϕ is a homomorphism so we just need to see that $\operatorname{Ker}\phi$ is trivial. However, if $[\alpha] \in \operatorname{Ker}\phi$ then $p_1 \circ \alpha \sim \gamma_{x_0}$ via some homotopy F and $p_2 \circ \alpha \sim \gamma_{y_0}$ via some homotopy G. Hence the map $H : I \times I \to X \times Y$ given by

$$H(t, s) = (F(t, s), G(t, s))$$

is a homotopy such that $\alpha \sim \gamma_{(x_0, y_0)}$ via H. Thus $[\alpha] = 1_{(x_0, y_0)}$ and we are done. □

Corollary 2.25. *$\pi_1(T^2) \cong \mathbb{Z} \times \mathbb{Z}$.*

Proof. This is straight from Theorems 2.17 and 2.24 together with the observation[4] that $T^2 \cong S^1 \times S^1$. □

2.5 Two Applications

We now apply our hard-won knowledge of $\pi_1(S^1)$ to prove two important theorems.

Fundamental Theorem of Algebra. *A non-constant polynomial with complex coefficients has a complex root.*

Proof. Seeking a contradiction, suppose that there exists a non-constant never-vanishing polynomial $p : \mathbb{C} \to \mathbb{C}$ which, with loss of generality, we may take to be monic[5]. Thus

$$p(z) = z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 \neq 0,$$

for all $z \in \mathbb{C}$.

Now define $f : S^1 \to S^1$ by

$$f(z) = \frac{p(z)}{|p(z)|} \in S^1,$$

for $z \in S^1$. f is well-defined and continuous since $p(z) \neq 0$, for all $z \in \mathbb{C}$.

First we show that f is homotopic to a constant map: indeed, define $F : S^1 \times I \to S^1$ by

$$F(z, s) = \frac{p(sz)}{|p(sz)|}$$

to get a homotopy from the constant map $c : z \mapsto p(0)/|p(0)|$ to f.

[4]Indeed, view the torus as $I \times I$ with opposite edges identified. Then $[t, s] \mapsto (e^{2\pi t}, e^{2\pi i s})$ is a well-defined homeomorphism from T^2 to $S^1 \times S^1$.

[5]Otherwise, divide through by the leading coefficient.

Now let $g : S^1 \to S^1$ be given by $g(z) = z^n$. We show that $f \simeq g$ also. For this, start with $\hat{F} : S^1 \times I \to \mathbb{C}$ given by

$$\hat{F}(z, s) = z^n + sa_{n-1}z^{n-1} + \cdots + s^n a_0,$$

which is certainly continuous, and satisfies

$$\hat{F}(z, 1) = p(z), \quad \hat{F}(z, 0) = z^n,$$

for all $z \in S^1$. Moreover, for $s \in (0, 1]$,

$$\hat{F}(z, s) = s^n p(z/s) \neq 0$$

while $\hat{F}(z, 0) = z^n \neq 0$, for all $z \in S^1$. Thus $\hat{F}$ is never zero and we define $F := \hat{F}/|\hat{F}| : S^1 \times I \to S^1$ to get a homotopy from g to f.

We conclude that $g \simeq c$ and this will rapidly lead to the contradiction we seek. Theorem 2.14 tells us that we have a commuting diagram

$$\begin{array}{ccc} \pi_1(S^1, 1) & \xrightarrow{g_*} & \pi_1(S^1, 1) \\ & \searrow^{c_*} & \downarrow{\scriptstyle [\gamma]_*} \\ & & \pi_1(S^1, c(1)) \end{array}$$

where $[\gamma]_*$ is the isomorphism induced by some path from $g(1) = 1$ to $c(1)$. Now c is constant so c_* is constant also: $c_*[\sigma] = 1_{c(1)}$, for all $[\sigma] \in \pi_1(S^1, 1)$. It follows that g_* is constant also so that $g \circ \sigma \sim \gamma_1$, for all loops σ. But taking $\sigma(t) = e^{2\pi it}$, we have $g \circ \sigma(t) = e^{2\pi int}$ so that $g \circ \sigma \not\sim \gamma_1$: indeed, for the isomorphism χ on Theorem 2.17, we have $\chi([g \circ \sigma]) = n \neq 0$. Thus we have arrived at a contradiction and the theorem is proved. □

For the next theorem, we need an easy lemma on retracts:

Lemma 2.26. *Let $A \subseteq X$ be a retract of X with retraction $r : X \to A$. Then, $r_* : \pi_1(X, a) \to \pi_1(A, a)$ is surjective, for any $a \in A$.*

Proof. Let $i : A \to X$ be the inclusion map $A \to X$. Then $r \circ i = \mathrm{id}_A : A \to A$, so that

$$r_* \circ i_* = \mathrm{id}_{\pi_1(A,a)},$$

which last is, of course, a bijection. Thus r_* surjects (and i_* injects). □

Brouwer Fixed Point Theorem. *Let $D^n = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$ be the closed unit disc[6] in $\mathbb{R}^n$ and let $f : D^n \to D^n$ be continuous.*

Then f has a fixed point: there is $x_0 \in D^n$ such that $f(x_0) = x_0$.

Proof. We can only give a complete proof when $n = 1$ or 2. However, the first part of the proof works in general.

We argue by contradiction and suppose that there is a continuous map $f : D^n \to D^n$ without fixed points: thus $f(x) \neq x$, for all $x \in D^n$. We use f to construct a retraction r from D^n to its boundary S^{n-1}: for each $x \in D^n$, draw the half-line from $f(x)$ through x and let $r(x)$ be the point where this half-line hits the boundary of the disc, see Figure 2.3. Then r is continuous since f is (exercise!) and, clearly, $r(x) = x$ when $x \in \partial D^n$. We therefore have the desired retraction.

Now for the part of the proof that depends on n: for $n = 1$, there is no continuous surjection, let alone a retraction, $r : D^1 = [-1, 1] \to S^0 = \{\pm 1\}$ since the domain is connected and the target disconnected.

When $n = 2$, Lemma 2.26 says that $r_* : \pi_1(D^2) \to \pi_1(S^1)$ surjects but, since D^2 is contractible, $\pi_1(D^2)$ is trivial while $\pi_1(S^1) \cong \mathbb{Z}$: a palpable contradiction. □

Remark. For the general case, we need functors that distinguish D^n and S^{n-1}. The higher homotopy groups π_k, $k \geq 2$, will do the job: $\pi_n(D^n)$ is trivial while $\pi_n(S^n) \cong \mathbb{Z}$.

[6] Here we use the Euclidean norm but, in fact, any norm will do.

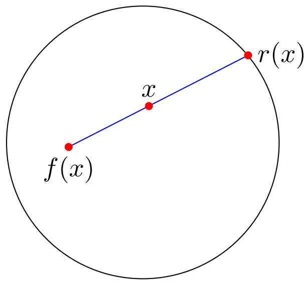


Figure 2.3: A retraction $r : D^n \to S^{n-1}$

Chapter 3

Covering Spaces

In this chapter, we abstract the essential properties of the map $\phi : \mathbb{R} \to S^1$ which was our main tool in the computation of the fundamental group $\pi_1(S^1, 1)$. This will give us a new perspective on the fundamental group in general.

3.1 Covering spaces and lifting theorems

Definition. A *covering map* $p : E \to X$ is a continuous surjection such that each $x \in X$ has an open neighbourhood U for which $p^{-1}(U)$ is a union of *disjoint* open sets S_i, $i \in I$, with $p|_{S_i} : S_i \to U$ a homeomorphism for each $i \in I$.

In this case, each such set U is said to be *evenly covered*, and the S_i, $i \in I$, are called the *sheets* over U.

A topological space E is said to be a *covering space* of X if there exists a covering map $p : E \to X$.

Examples.

1. The map $\phi : \mathbb{R} \to S^1$ is a covering map: the open sets $U := S^1 \setminus \{-1\}$ and $V := S^1 \setminus \{1\}$ are both evenly covered. The sheets over U are the sets $(n - \frac{1}{2}, n + \frac{1}{2})$, $n \in \mathbb{Z}$, and the sheets over V are the sets $(k, k+1)$, $k \in \mathbb{Z}$.
2. Consider the map $\pi_2 : \mathbb{Z} \times X \to X$ defined by $\pi_2(n, x) = x$, where $\mathbb{Z}$ has the discrete topology and $\mathbb{Z} \times X$ the product topology. Then π_2 is a covering map: X itself is evenly covered with sheets $\{n\} \times X$, $n \in \mathbb{Z}$.

The basic properties of covering maps are contained in the following:

Exercises. Let $p : E \to X$ be a covering map.

1. For $x \in X$, the *fibre over* x, $p^{-1}(\{x\})$ is discrete (that is, the induced topology from E is the discrete topology).
2. The map $p : E \to X$ is a *local homeomorphism*: that is, each $e \in E$ has an open neighbourhood S such that $p(S)$ is open in X and $p_{|S} : S \to p(S)$ is a homeomorphism.
3. The map $p : E \to X$ is open, whence X has the quotient topology induced by p.

In particular E and X have the same local properties: for example, E is locally path-connected/locally compact/is a manifold if and only if X is.

Lifts

The main property of the map $\phi : \mathbb{R} \to S^1$ that we used was the possibility of uniquely lifting paths and homotopies in S^1 to paths and homotopies in $\mathbb{R}$. The same thing works in our more general setting.

Definition. Let $p : E \to X$ be a covering map and $f : Y \to X$ a continuous map of topological spaces. A *lift of f (with respect to p)* is a continuous map $f' : Y \to E$ such that $p \circ f' = f$. Thus f' makes the following diagram commute:

$$\begin{array}{ccc} & & E \\ & \overset{f'}{\nearrow} & \big\downarrow p \\ Y & \xrightarrow{\;f\;} & X \end{array}$$

The two big questions about lifts are:

1. For what Y and f does a lift f' exist?
2. How unique is f' when it does exist?

We shall give fairly complete answers to both questions but tackle the uniqueness issue first:

Lemma 3.1 (Unique Lifting Property). *Let Y be connected, $p : E \to X$ be a covering map, and $f : Y \to X$ continuous. Suppose that $f_1, f_2 : Y \to E$ are two lifts of f, that is, $p \circ f_1 = p \circ f_2 = f$. Then*

$$A := \{y \in Y : f_1(y) = f_2(y)\}$$

is either $\emptyset$ or Y.

Thus, $f_1 = f_2$ as soon as they agree at a single point.

Proof. Set $D = Y \setminus A$. We will show that both A and D are open so that one of them must be empty since Y is connected.

For this, fix $y \in Y$ and let U be an evenly covered neighbourhood of $f(y) \in X$. For $i = 1, 2$, let S_i be the sheet over U containing $f_i(y)$, and set

$$V = f_1^{-1}(S_1) \cap f_2^{-1}(S_2),$$

which is an open neighbourhood of $y \in Y$. We will show that either $V \subset A$ or $V \subset D$ from which it immediately follows that both A and D are open. Let $z \in V$ so that $f_i(z) \in S_i$, $i = 1, 2$. There are two cases to consider:

1. If $y \in A$ then $f_1(y) = f_2(y)$ so that $S_1 = S_2$. Then both $f_i(z) \in S_1$ and $p(f_1(z)) = f(z) = p(f_2(z))$ so that $f_1(z) = f_2(z)$ since $p_{|S_1}$ injects. Thus $z \in A$ and so $V \subset A$.
2. If $y \in D$ then $f_1(y) \neq f_2(y)$ so that $S_1 \neq S_2$ (otherwise $p(f_1(y)) = p(f_2(y))$ would force a contradiction). Thus $S_1 \cap S_2 = \emptyset$ and so $f_1(z) \neq f_2(z)$. This means that $z \in D$ whence $V \subset D$.

□

Since lifts are determined by their value at a single point, the following notation will be useful:

Notation. Write $f : (X, x) \to (Y, y)$ if $f : X \to Y$, $x \in X$, $y \in Y$ and $f(x) = y$.

Theorem 3.2 (Path-Lifting Theorem). *Let $p : (E, e_0) \to (X, x_0)$ be a covering map and $\sigma : (I, 0) \to (X, x_0)$ a path, starting at $x_0 \in X$. Then there exists a unique lift $\sigma'_{e_0} : (I, 0) \to (E, e_0)$:*

$$\begin{array}{ccc} & & (E, e_0) \\ & \overset{\sigma'_{e_0}}{\nearrow} & \big\downarrow p \\ (I, 0) & \xrightarrow{\;\sigma\;} & (X, x_0) \end{array}$$

Proof. The uniqueness is immediate from Theorem 3.1.

X has a cover by evenly covered open sets, so by the Lebesgue Covering Lemma, there is a partition

$$0 = t_0 < t_1 < \cdots < t_N = 1,$$

with each $\sigma([t_{i-1}, t_i])$ lying in an evenly covered set.

Subcase: *let $e_i \in p^{-1}(\{\sigma(t_i)\})$. Then there is a lift $\tilde{\sigma}_{e_i} : ([t_i, t_{i+1}], t_i) \to (E, e_i)$ of $\sigma|_{[t_i,t_{i+1}]}$.*

Proof. Let $U \subset X$ be an evenly covered open set containing $\sigma([t_i, t_{i+1}])$ and S the sheet over U containing e_i. Then $p_{|S} : S \to U$ is a homeomorphism with inverse ψ and we set

$$\tilde{\sigma}_{e_i} = \psi \circ \sigma|_{[t_i,t_{i+1}]}.$$

□

We now construct σ'_{e_0} by induction. Our induction hypothesis is that there exists a continuous map $\sigma'_i : ([0, t_i], 0) \to (E, e_0)$ such that

$$p \circ \sigma'_i = \sigma|_{[0,t_i]}.$$

For the base case $i = 1$, take $\sigma'_1 = \tilde{\sigma}_{e_0}$.

For the induction step, assume that the result holds in the case i, for some $i \in \{1, \dots, N-1\}$, and define $\sigma'_{i+1} : ([0, t_{i+1}], 0) \to (E, e_0)$ by

$$\sigma'_{i+1}(t) = \begin{cases} \sigma'_i(t), & t \in [0, t_i] \\ \tilde{\sigma}_{\sigma'_i(t_i)}(t), & t \in [t_i, t_{i+1}]. \end{cases}$$

Then σ'_{i+1} is well-defined since both definitions agree at t_i and so is continuous by Lemma 2.1. We may therefore induct and define $\sigma'_{e_0} = \sigma'_N$. □

The uniqueness part of (3.2) gives us quite a lot of information:

- for $\sigma' : I \to E$, $(p \circ \sigma')'_{\sigma'(0)} = \sigma'$;
- $(\gamma_{x_0})'_{e_0} = \gamma_{e_0}$ so that constant paths lift to constant paths;
- if $\sigma, \tau : I \to X$ are paths with $\sigma(0) = x_0$ and $\tau(0) = \sigma(1)$, then

  $$(\sigma \cdot \tau)'_{e_0} = \sigma'_{e_0} \cdot \tau'_{\sigma'_{e_0}(1)}$$

  so that we know how to construct lifts of products of paths.

We prove the based homotopies lift in two stages: first we construct a lift and then we convince ourselves that the lift is again a based homotopy.

Theorem 3.3 (Homotopy Lifting I)**.** *Let $p : (E, e_0) \to (X, x_0)$ be a covering map and $F : (I \times I, 0) \to (X, x_0)$ a continuous map. Then there is a unique lift $F'_{e_0} : (I \times I, 0) \to (E, e_0)$:*

$$\begin{array}{ccc} & & (E, e_0) \\ & \overset{F'_{e_0}}{\nearrow} & \big\downarrow p \\ (I \times I, 0) & \xrightarrow{F} & (X, x_0) \end{array}$$

Proof. Again we get the uniqueness part from (3.1).

The Lebesgue Covering Lemma gives us partitions

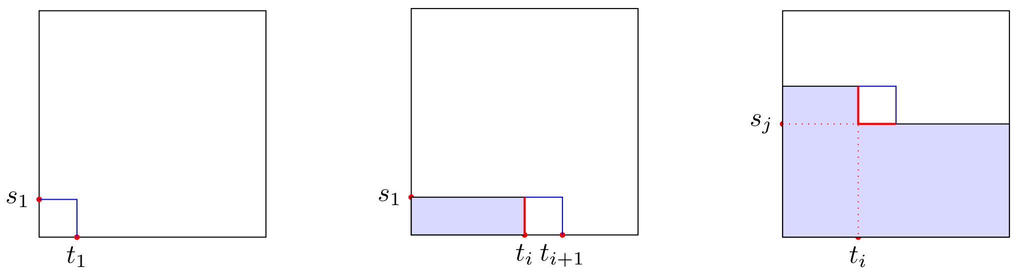
$$0 = t_0 < t_1 < \cdots < t_N = 1, \quad 0 = s_0 < s_1 < \cdots < s_M = 1,$$

such that each $F([t_i, t_{i+1}] \times [s_j, s_{j+1}])$ lies in an evenly covered open subset of X.

Subcase: *For* $e \in p^{-1}(\{F(t_i, s_j)\})$*, there is a lift* $\tilde{F}_e : ([t_i, t_{i+1}] \times [s_j, s_{j+1}], (t_i, s_j)) \to (E, e)$ *of* $F|_{[t_i,t_{i+1}]\times[s_j,s_{j+1}]}$.

Proof. Just like the subcase of (3.2). ☐

We now construct F'_{e_0} a rectangle at a time.



(a) The first brick. (b) The first row. (c) Carrying on.

Figure 3.1: Bricklaying

1. Take $e = e_0$ in the subcase to define $F'_{e_0} = \tilde{F}_e$ on $[0, t_1] \times [0, s_1]$, shown in Figure 3.1a.
2. Now suppose that that F'_{e_0} has been defined on $[0, t_i] \times [0, s_1]$, the shaded area in Figure 3.1b. We use the subcase above with $e = F'_{e_0}(t_i, 0)$ to define $F'_{e_0} = \tilde{F}_e$ on $[t_i, t_{i+1}] \times [0, s_1]$. To see that this is well-defined (and hence continuous by Lemma 2.1), we must check that our definitions agree on $\{t_i\} \times [0, s_1]$ (shown in red in Figure 3.1b). However, both definitions give lifts of $F|_{\{t_i\}\times[0,s_1]}$ that agree at $(t_i, 0)$ and so agree everywhere by (3.1).
3. Finally, suppose that we have defined F'_{e_0} on the shaded area in Figure 3.1c. Use the subcase with $e = F'_{e_0}(t_i, s_j)$ to define $F'_{e_0} = \tilde{F}_e$ on $[t_i, t_{i+1}] \times [s_j, s_{j+1}]$. Again, we must check that F'_{e_0} is well-defined, and so continuous, and this means seeing that the definitions of F'_{e_0} agree on

$$J := ([t_i, t_{i+1}] \times \{s_j\}) \cup (\{t_i\} \times [s_j, s_{j+1}]),$$

   or, if $i = 0$, on $J = [0, t_1] \times [s_j, s_{j+1}]$. However, in either case, J is connected and our competing definitions lift $F_{|J}$ while agreeing at (t_i, s_j) and so agree on all of J by (3.1).

Now induct to define F'_{e_0} on all of $I \times I$. ☐

Applications

Corollary 3.4 (Homotopy Lifting II)**.** *Let* $p : (E, e_0) \to (X, x_0)$ *be a covering map. If* $\sigma, \tau : (I, 0) \to (X, x_0)$ *are continuous and* $\sigma \sim \tau$*, then* $\sigma'_{e_0} \sim \tau'_{e_0}$.

In particular, $\sigma'_{e_0}(1) = \tau'_{e_0}(1)$.

Proof. Let $\sigma \sim \tau$ via a homotopy $F : I \times I \to X$, and set $F' = F'_{e_0}$, the lift of F from Theorem 3.3. Define paths $\alpha, \beta, \gamma, \delta : I \to E$ by

$$\alpha(s) = F'(0, s), \quad \beta(s) = F'(1, s), \quad \gamma(t) = F'(t, 0), \quad \delta(t) = F'(t, 1), \quad \forall s, t \in I.$$

Then:

1. $(p \circ \gamma)(t) = F(t, 0) = \sigma(t)$, so γ is a lift of σ, and $\gamma(0) = F'(0, 0) = e_0$, so $\gamma = \sigma'_{e_0}$.

2. Similarly, $(p \circ \alpha)(s) = F(0, s) = x_0$, $\forall s \in I$, so α lifts the constant path γ_{x_0} and $\alpha(0) = e_0$, so $\alpha = \gamma_{e_0}$.

3. In particular, $\delta(0) = \alpha(1) = e_0$, while $(p \circ \delta)(t) = \tau(t)$ for all $t \in I$, so $\delta = \tau'_{e_0}$.

4. Finally, $(p \circ \beta)(s) = F(1, s) = \sigma(1)$, $\forall s \in I$, so β lifts a constant path and so is constant with value $\beta(0) = \gamma(1) = \sigma'_{e_0}(1)$.

Hence $\sigma'_{e_0} \sim \tau'_{e_0}$ via $F' : I \times I \to E$. □

Corollary 3.5. *Let $p : (E, e_0) \to (X, x_0)$ be a covering map. Then $p_* : \pi_1(E, e_0) \to \pi_1(X, x_0)$ injects.*

Proof. Let $\alpha : I \to E$ be a loop in E based at e_0 such that $[\alpha] \in \operatorname{Ker} p_*$. Hence $p \circ \alpha \sim \gamma_{x_0}$. Now $(p \circ \alpha)'_{e_0} = \alpha$ while $(\gamma_{x_0})'_{e_0} = \gamma_{e_0}$ so that Corollary 3.4 gives $\alpha \sim \gamma_{e_0}$. Thus $\ker p_*$ is trivial whence p_* injects. □

Exercise (A little challenging)**.** Let E be path-connected. Then $p_* : \pi_1(E, e_0) \to \pi_1(X, x_0)$ is an isomorphism if and only if $p : (E, e_0) \to (X, x_0)$ is a homeomorphism.

Let us now recall a little algebra:

Definition. A *(right) action of a group G on a set A* is a map $A \times G \to A$, written $(a, g) \mapsto a \cdot g$, such that

$$a \cdot 1 = a,$$
$$a \cdot (g_1 g_2) = (a \cdot g_1) \cdot g_2,$$

for all $a \in A$, $g_1, g_2 \in G$.

Now let $p : E \to X$ be a covering map, $x_0 \in X$ and consider the fibre of p over $x_0 : p^{-1}\{x_0\}$. We are going to define an action of $\pi_1(X, x_0)$ on $p^{-1}\{x_0\}$. For this, let $e \in p^{-1}(\{x_0\})$ and $[\sigma] \in \pi_1(X, x_0)$, and define $e \cdot [\sigma] \in p^{-1}\{x_0\}$ by

$$e \cdot [\sigma] = \sigma'_e(1)$$

which does indeed lie in $p^{-1}\{x_0\}$, since $p(\sigma'_e(1)) = \sigma(1) = x_0$. We note:

1. This is well-defined: if $[\sigma] = [\tau]$, then $\sigma \sim \tau$ and thus $\sigma'_e(1) = \tau'_e(1)$ by Corollary 3.4.

2. This is an action:

   (a) $e \cdot 1_{x_0} = e \cdot [\gamma_{x_0}] = (\gamma_{x_0})'_e(1)$. But $(\gamma_{x_0})'_e = \gamma_e$ so that $e \cdot 1_{x_0} = \gamma_e(1) = e$, for all $e \in p^{-1}\{x_0\}$.

   (b) For $[\sigma], [\tau] \in \pi_1(X, x_0)$ and $e \in p^{-1}\{x_0\}$, we have $(\sigma \cdot \tau)'_e = \sigma'_e \cdot \tau'_{\sigma'_e(1)}$ so that

$$e \cdot ([\sigma] \cdot [\tau]) = (\sigma \cdot \tau)'_e(1) = \tau'_{\sigma'_e(1)}(1) = (\sigma'_e(1)) \cdot [\tau] = (e \cdot [\sigma]) \cdot [\tau].$$

Exercises.

(i) If E is path-connected, then the action is *transitive*: if $e_1, e_2 \in p^{-1}(\{x_0\})$, then there is $[\sigma] \in \pi_1(X, x_0)$ such that $e_1 \cdot [\sigma] = e_2$.

(ii) The *stabiliser* of $e \in p^{-1}(\{x_0\})$ (that is, the subgroup $\{[\sigma] \in \pi_1(X, x_0) : e \cdot [\sigma] = e\}$) is $p_*(\pi_1(E, e))$.

(iii) (Algebra) Let A be a set with a transitive right G-action with trivial stabilisers: $a \cdot g = a$, for some $a \in A$, if and only if $g = 1$. Then, for fixed $a \in A$, the map $G \to A : g \mapsto a \cdot g$ is a bijection. In particular:

(iv) If E is simply connected and $e \in p^{-1}\{x_0\}$, then the map $\mu_e : \pi_1(X, x_0) \to p^{-1}(\{x_0\})$ defined by

$$\mu_e([\sigma]) = e \cdot [\sigma]$$

is a bijection.

Here is an application of these ideas: we will soon see that the quotient map $p : S^n \to \mathbb{R}P^n$ is a covering map and, when $n \geq 2$, we know from Theorem 2.23 that S^n is simply connected. We therefore deduce that there is a bijection between $\pi_1(\mathbb{R}P^n, [x])$ and $p^{-1}\{x\} = \{\pm x\}$. Thus $|\pi_1(\mathbb{R}P^n)| = 2$. However, there is only one group with 2 elements so that $\pi_1(\mathbb{R}P^n) \cong \mathbb{Z}/2\mathbb{Z}$. We shall see a slightly more refined argument in Theorem 3.10 below.

Now let us return to the existence question for lifts and give a complete answer under very mild hypotheses on the domain:

Theorem 3.6 (Ultimate Lifting Theorem)**.** *Let $p : (E, e_0) \to (X, x_0)$ be a covering map, Y a connected and locally path-connected topological space and $f : (Y, y_0) \to (X, x_0)$ a continuous map. Then f has a unique*[1] *lift $f' : (Y, y_0) \to (E, e_0)$ if and only if*

$$f_*(\pi_1(Y, y_0)) \subset p_*(\pi_1(E, e_0)).$$

Proof. The forward implication is easy: if f' exists, then $p \circ f' = f$. Hence $f_* = p_* \circ f'_*$ and, in particular,

$$f_*(\pi_1(Y, y_0)) = p_*(f'_*(\pi_1(Y, y_0))) \subset p_*(\pi_1(E, e_0)).$$

The converse is more challenging: somehow we must cook up the map f' (and then show it is continuous). To arrive at a definition, we begin by imagining that f' *does* exist. In that case, if τ was a path in Y from y_0 to y, then $f' \circ \tau$ would start at e_0 and $p \circ f' \circ \tau = f \circ \tau$, so that $(f \circ \tau)'_{e_0} = f' \circ \tau$. In particular,

$$f'(y) = (f' \circ \tau)(1) = (f \circ \tau)'_{e_0}(1).$$

Now stop imagining and notice that this gives us a candidate definition for $f'(y)$: *define* $f'(y)$ to be $(f \circ \tau)'_{e_0}(1)$, where τ is some path from y_0 to y (such a path exists for all $y \in Y$ since Y is path-connected). Of course, there are issues with this definition which we need to resolve:

(a) $f'(y)$ is well-defined (that is, independent of the choice of τ);

(b) f' is continuous.

We do (a) first. If τ_1 and τ_2 are paths from y_0 to y, and $\tau_1 \sim \tau_2$, then $f \circ \tau_1 \sim f \circ \tau_2$ whence $(f \circ \tau_1)'_{e_0} \sim (f \circ \tau_2)'_{e_0}$, and in particular

$$(f \circ \tau_1)'_{e_0}(1) = (f \circ \tau_2)'_{e_0}(1).$$

More generally, for any paths τ_1, τ_2 from y_0 to y, we have

$$\tau_1 \sim (\tau_1 \cdot \tau_2^{-1}) \cdot \tau_2 = \sigma \cdot \tau_2,$$

with $\sigma = \tau_1 \cdot \tau_2^{-1}$ a loop at y_0. Now,

$$f \circ (\sigma \cdot \tau_2) = (f \circ \sigma) \cdot (f \circ \tau_2),$$

so

$$(f \circ (\sigma \cdot \tau_2))'_{e_0} = (f \circ \sigma)'_{e_0} \cdot (f \circ \tau_2)'_e,$$

where $e = (f \circ \sigma)'_{e_0}(1)$. In fact $e = e_0$: by hypothesis,

$$f_*([\sigma]) = p_*([\sigma']),$$

[1] By Theorem 3.1!

for some loop σ' based at e_0. Therefore $f \circ \sigma \sim p \circ \sigma'$, and taking lifts gives

$$(f \circ \sigma)'_{e_0} \sim (p \circ \sigma')'_{e_0} = \sigma',$$

so that

$$e = (f \circ \sigma)'_{e_0}(1) = \sigma'(1) = e_0.$$

Therefore, from the first part,

$$(f \circ \tau_1)'_{e_0}(1) = (f \circ (\sigma \cdot \tau_2))'_{e_0}(1) = (f \circ \tau_2)'_{e=e_0}(1),$$

so that f' is well-defined.

Moreover, by construction,

$$(p \circ f')(y) = (f \circ \tau)(1) = f(y),$$

so $p \circ f' = f$. Finally, take $\tau = \gamma_{y_0}$ to see that $f'(y_0) = e_0$.

To show that f' is continuous, first note that E has a base of open sets on each of which p is a homeomorphism. Let S be one such set—it suffices to show that $(f')^{-1}(S)$ is open in Y. So let $U = p(S) \subset X$, which is open in X, and let $\psi : U \to S$ be the inverse of $p|_S$. Let $y \in (f')^{-1}(S)$; it suffices to find a neighbourhood V of y contained in $(f')^{-1}(S)$. Now $f(y) \in U$ so $y \in f^{-1}(U)$, and $f^{-1}(U)$ is open by the continuity of f. Therefore, since Y is locally path-connected, there is a path-connected neighbourhood V of y with $V \subset f^{-1}(U)$. We claim that $V \subset (f')^{-1}(S)$. Indeed, let $y_1 \in V$ and choose a path τ_1 in V from y to y_1, and also a path τ in Y from y_0 to y. Then $\tau \cdot \tau_1$ is a path from y_0 to y_1 whence

$$f'(y_1) = (f \circ (\tau \cdot \tau_1))'_{e_0}(1) = (f \circ \tau)'_{e_0} \cdot (f \circ \tau_1)'_{f'(y)}(1),$$

since $f'(y) = (f \circ \tau)'_{e_0}(1)$. But $f \circ \tau_1$ is a path in U (since $V \subset f^{-1}U$) so that $\psi \circ f \circ \tau_1$ is defined and

$$(\psi \circ f \circ \tau_1)(0) = \psi(f(y)) = f'(y), \quad p \circ \psi \circ f \circ \tau_1 = f \circ \tau_1,$$

so that

$$\psi \circ f \circ \tau_1 = (f \circ \tau_1)'_{f'(y)},$$

whence

$$f'(y_1) = (f \circ \tau_1)'_{f'(y)}(1) = (\psi \circ f \circ \tau_1)(1) = \psi(f(y_1)) \in S$$

and we are done. □

Of course, the condition on $\pi_1(Y)$ is vacuously satisfied if Y is simply connected:

Corollary 3.7. *If Y is simply connected and locally path-connected, then any continuous function $f : (Y, y_0) \to (X, x_0)$ has a lift $f' : (Y, y_0) \to (E, e_0)$.*

3.2 The Fundamental Group and Deck Translations

Definition. Let $p : (E, e_0) \to (X, x_0)$ be a covering map. A *deck translation of* p is a homeomorphism $\phi : E \to E$ such that $p \circ \phi = p$. Thus:

$$\begin{array}{ccc} E & \xrightarrow{\phi} & E \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle p} & \\ & X & \end{array}$$

Remark. Observe that deck translations are lifts of p with respect to p! We shall use this observation several times.

Exercises.

1. What are the deck translations of $\phi : \mathbb{R} \to S^1$?

2. Show that deck translations are a group under composition of maps.

The usefulness of this concept comes from the fact that the fundamental group is isomorphic to the group of deck translations of a simply connected covering space. This gives a quite different viewpoint on the fundamental group as it realises it as a group of symmetries of a concrete space rather than some abstract collection of based homotopies of paths.

Theorem 3.8. *Let E be simply connected and locally path-connected and $p : (E, e_0) \to (X, x_0)$ a covering map. Let G be the group of deck translations of p. Then $G \cong \pi_1(X, x_0)$.*

Proof. Recall that, since E is simply connected, any paths σ_1, σ_2 in E with the same endpoints satisfy $\sigma_1 \sim \sigma_2$. We use this to define a map $\chi : G \to \pi_1(X, x_0)$ as follows. For $\phi \in G$, let σ' be a path from e_0 to $\phi(e_0)$ (remember that E is path-connected!). Then $p(\phi(e_0)) = p(e_0) = x_0$ so that $p \circ \sigma'$ is a loop at x_0 and we set

$$\chi(\phi) = [p \circ \sigma'] \in \pi_1(X, x_0).$$

This is well-defined, because if σ'' is another path from e_0 to $\phi(e_0)$, then $\sigma' \sim \sigma''$ so $p \circ \sigma' \sim p \circ \sigma''$.

We now show that χ is an isomorphism of groups.

χ **is a homomorphism of groups.** Let $\phi_1, \phi_2 \in G$ and let σ', τ' be paths from e_0 to $\phi_1(e_0)$, $\phi_2(e_0)$ respectively. Then $\phi_1 \circ \tau'$ is a path from $\phi_1(e_0)$ to $(\phi_1 \circ \phi_2)(e_0)$, Therefore, $\sigma' \cdot (\phi_1 \circ \tau')$ is a path from e_0 to $(\phi_1 \circ \phi_2)(e_0)$ whence

$$\chi(\phi_1 \circ \phi_2) = [p \circ (\sigma' \cdot (\phi_1 \circ \tau'))] = [p \circ \sigma'] \cdot [p \circ \phi_1 \circ \tau'] = [p \circ \sigma'] \cdot [p \circ \tau'] = \chi(\phi_1) \cdot \chi(\phi_2).$$

χ **is injective.** We show that $\operatorname{Ker} \chi$ is trivial. If $\phi \in \operatorname{Ker} \chi$ and σ' is a path from e_0 to $\phi(e_0)$, then $[p \circ \sigma'] = 1_{x_0}$ so that $p \circ \sigma' \sim \gamma_{x_0}$. Now unique lifting tells us that $\sigma' \sim \gamma_{e_0}$ and, in particular, $\sigma'(1) = e_0 = \phi(e_0)$. Since both ϕ and id_E lift p (with respect to itself) and agree at e_0, unique lifting tells us $\phi = \mathrm{id}_E$.

χ **is surjective.** Let $[\sigma] \in \pi_1(X, x_0)$. We seek $\phi \in G$ such that $\phi(e_0) = \sigma'_{e_0}(1)$ for then $\chi(\phi) = [p \circ \sigma'_{e_0}] = [\sigma]$.

So let $e_1 = \sigma'_{e_0}(1) \in p^{-1}(\{x_0\})$. Theorem 3.7 gives us continuous maps $\phi : (E, e_0) \to (E, e_1)$ and $\psi : (E, e_1) \to (E, e_0)$ lifting p. Then $\psi \circ \phi, \mathrm{id}_E : (E, e_0) \to (E, e_0)$ both lift p and agree at e_0 so that, by (3.1) $\psi \circ \phi = \mathrm{id}_E$. Similarly, $\phi \circ \psi = \mathrm{id}_E$ so that ϕ is a homeomorphism and thus a deck translation with $\phi(e_0) = e_1$.

□

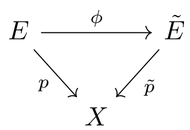
Exercise. Recall the right action of $\pi_1(X, x_0)$ on $p^{-1}(\{x_0\})$. Show that $e_0 \cdot \chi(\phi) = \phi(e_0)$.

3.3 Universal Covers

Theorem 3.8 tells us that simply connected covering spaces are useful objects so let us christen them:

Definition. A topological space E is a *universal cover of X* if E is simply connected and there is a covering map $p : E \to X$.

Proposition 3.9 (Uniqueness of Universal Covers)**.** *If $p : E \to X$ and $\tilde{p} : \tilde{E} \to X$ are both universal covers of a locally path-connected space X, then there is a homeomorphism $\phi : E \to \tilde{E}$ such that $\tilde{p} \circ \phi = p$. Thus:*



Proof. First note that both $E, \tilde{E}$ are locally path-connected, since X is, so that the Ultimate Lifting Theorem is available.

Fix $x_0 \in X$, $e_0 \in p^{-1}(\{x_0\})$ and $\tilde{e}_0 \in \tilde{p}^{-1}(\{x_0\})$; then the ultimate lifting theorem 3.7 yields lifts $\phi : (E, e_0) \to (\tilde{E}, \tilde{e}_0)$ and $\psi : (\tilde{E}, \tilde{e}_0) \to (E, e_0)$ of p with respect to $\tilde{p}$ and $\tilde{p}$ with respect to p respectively. Thus $\tilde{p} \circ \phi = p$.

Moreover, both $\psi \circ \phi$ and id_E lift p (with respect to itself) and agree at e_0 and so everywhere. Similarly $\phi \circ \psi = \mathrm{id}_{\tilde{E}}$ so that ϕ is a homeomorphism. □

When does a space X have a universal cover? There is a necessary condition:

Definition. A topological space X is said to be *semi-locally simply connected* if each point $x \in X$ has a neighbourhood U with the property that any loop in U is based homotopic in X to the constant loop.

Exercise. If X has a universal cover, then X is semi-locally simply connected.

For any decent topological space, the converse is true also:

Theorem. *If X is connected, locally path-connected and semi-locally simply connected then X has a universal cover.*

We shall not prove this theorem but content ourselves with seeing how E might be constructed. Again, we imagine the situation when a universal cover E does exist with covering map $p : (E, e_0) \to (X, x_0)$. Let $e \in E$. Then, since E is path-connected, there is a path τ' from e_0 to e and so a path $p \circ \tau'$ from x_0 to $p(e)$. Moreover, if τ'' was another such path then $\tau' \sim \tau''$, since E is simply connected, so that $p \circ \tau' \sim p \circ \tau''$.

We have therefore constructed a map $e \mapsto [p \circ \tau']$ from E to the set of based homotopy classes of paths starting at x_0. In fact, this map is a bijection with inverse $[\tau] \mapsto \tau'_{e_0}(1)$ (exercise!). We may therefore *define* E to be this set of based homotopy classes. In this setup, we take e_0 to be $[\gamma_{x_0}]$ and define $p : E \to X$ by $p[\tau] = \tau(1)$. Of course, there is much more to do to prove the theorem (topologise E so that p is a covering map; show that E is simply connected in that topology) but we shall go no further.

3.4 Topology of $\mathbb{R}P^n$ and the Borsuk–Ulam Theorem

Recall that $\mathbb{R}P^n$ is the topological quotient $S^n/_\sim$, where $x \sim y \iff x = \pm y$. Let $p : S^n \to \mathbb{R}P^n$ be the quotient map $x \mapsto [x]$.

Exercises.

1. $p : S^n \to \mathbb{R}P^n$ is a covering map.
2. The map $\phi : \mathbb{R}P^1 \to S^1$ given by $\phi([z]) = z^2$ is a well-defined homeomorphism.

With these in hand, we can compute $\pi_1(\mathbb{R}P^1)$.

Theorem 3.10.

1. $\pi_1(\mathbb{R}P^1) \cong \mathbb{Z}$.

2. $\pi_1(\mathbb{R}P^n) \cong \mathbb{Z}/2\mathbb{Z}$*, for* $n \geq 2$.

Proof. Since $\mathbb{R}P^1 \cong S^1$, $\pi_1(\mathbb{R}P^1) \cong \mathbb{Z}$ thanks to Theorem 2.17.

For $n \geq 2$, S^n is simply connected (and locally path-connected) and $p : S^n \to \mathbb{R}P^n$ is a covering map. Hence, by Theorem 3.8, $\pi_1(\mathbb{R}P^n) \cong G$, the group of deck translations of p. However

$$G = \{\mathrm{id}_{S^n}, -\mathrm{id}_{S^n}\}.$$

Indeed, it is certainly true that $\pm \mathrm{id}_{S^n} \in G$, and if $\phi : S^n \to S^n$ is a deck translation and $x_0 \in S^n$, then $\phi(x_0) = \pm x_0$ since $\phi(x_0) \in p^{-1}\{[x_0]\} = \{\pm x_0\}$. Thus since both ϕ and $\pm \mathrm{id}_{S^n}$ all lift p and ϕ coincides with one of the latter at x_0, (3.1) guarantees that $\phi = \pm \mathrm{id}_{S^n}$. □

Our main application is to prove the celebrated:

Theorem 3.11 (Borsuk–Ulam)**.** *There does not exist a continuous map $\phi : S^n \to S^{n-1}$ which is* antipode preserving*, that is, such that $\phi(-x) = -\phi(x)$, for all $x \in S^n$.*

Proof. For the usual reasons, we can only prove the results when $n = 1$ or 2.

If $n = 1$, note that S^1 is connected while $S^0 = \{-1, 1\}$ is disconnected. Thus any continuous map $\phi : S^1 \to S^0$ is constant and so not antipode preserving.

Now suppose that $n = 2$, and that there exists a continuous antipode preserving map $\phi : S^2 \to S^1$. Define $\psi : \mathbb{R}P^2 \to \mathbb{R}P^1$ by

$$\psi([x]) = [\phi(x)],$$

for all $[x] \in \mathbb{R}P^2$. ψ is well-defined since $\phi(-x) \sim \phi(x)$. We have a commuting diagram

$$\begin{array}{ccc} S^2 & \xrightarrow{\phi} & S^1 \\ \downarrow p & & \downarrow p \\ \mathbb{R}P^2 & \xrightarrow{\psi} & \mathbb{R}P^1, \end{array}$$

and the universal property of quotients (1.5) tells us that ψ is continuous because $p \circ \phi$ is.

Fix $x_0 \in S^2$, set $y_0 = \phi(x_0)$, and consider the homomorphism $\psi_* : \pi_1(\mathbb{R}P^2, [x_0]) \to \pi_1(\mathbb{R}P^1, [y_0])$. Theorem 3.10 tells us that $\pi_1(\mathbb{R}P^2, [x_0]) \cong \mathbb{Z}/2\mathbb{Z}$, a *finite* group, while $\pi_1(\mathbb{R}P^1, [y_0]) \cong \mathbb{Z}$. However, any group homomorphism from a finite group to $\mathbb{Z}$ is constant: all elements of a finite group have finite order and so do their images under a homomorphism but 0 is the only finite order element of $\mathbb{Z}$!

Thus, if σ is a loop in $\mathbb{R}P^2$ based at $[x_0]$, we have

$$\psi_*([\sigma]) = 1_{[y_0]},$$

or, equivalently, $\psi \circ \sigma \sim \gamma_{[y_0]}$.

Now we can get a contradiction. Let $\sigma' : I \to S^2$ be a path in S^2 from x_0 to $-x_0$, and let $\sigma = p \circ \sigma'$ so that σ is a loop at $[x_0] \in \mathbb{R}P^2$. Note that

$$\psi \circ \sigma = p \circ \phi \circ \sigma',$$

so that

$$p \circ \phi \circ \sigma' \sim \gamma_{[y_0]}.$$

Taking lifts starting at y_0 then yields:

$$\phi \circ \sigma' \sim \gamma_{y_0},$$

and, in particular, $\phi \circ \sigma'(1) = \gamma_{y_0}(1) = y_0$. But the antipode preserving property of ϕ gives

$$\phi \circ \sigma'(1) = \phi(-x_0) = -\phi(x_0) = -y_0 \neq y_0,$$

and we have arrived at a contradiction. □

Corollary 3.12. *If $f : S^n \to \mathbb{R}^n$ is continuous and* odd*, that is, $f(-x) = -f(x)$, for all $x \in S^n$, then there is $x_0 \in S^n$ such that $f(x_0) = 0$.*

Proof. If not, define $g : S^n \to S^{n-1}$ by $g(x) = f(x)/\|f(x)\|$ to get a continuous antipode preserving map in contradiction to the Borsuk–Ulam Theorem. □

Corollary 3.13. *If $f : S^n \to \mathbb{R}^n$ is continuous, then there is $x_0 \in S^n$ such that $f(x_0) = f(-x_0)$. (In particular, f does not inject.)*

Proof. Define $g : S^n \to \mathbb{R}^n$ by $g(x) = f(x) - f(-x)$, $\forall x \in S^n$. Then g is continuous and satisfies $g(-x) = -g(x)$, $\forall x \in S^n$, so that Corollary 3.12 applies. □

Theorem 3.14 (Ham Sandwich Theorem)**.** *Let $X_1, \ldots, X_n$ be bounded Lebesgue measurable subsets of $\mathbb{R}^n$. Then there exists a hyperplane bisecting each set X_i simultaneously: that is, each X_i has the same volume*[2] *on each side of the hyperplane.*

Sketch proof. Let $e_{n+1} = (0, 0, \ldots, 0, 1) \in \mathbb{R}^{n+1}$, and view $\mathbb{R}^n$ as the orthogonal complement of e_{n+1}. For $x \in S^n$, let π_x be the hyperplane in $\mathbb{R}^{n+1}$ orthogonal to x passing through e_{n+1} (see Figure 3.2). Then $\pi_x \cap \mathbb{R}^n$ is a hyperplane in $\mathbb{R}^n$ except when $x = e_{n+1}$, in which case $\pi_{e_{n+1}} \cap \mathbb{R}^n = \emptyset$. Note that

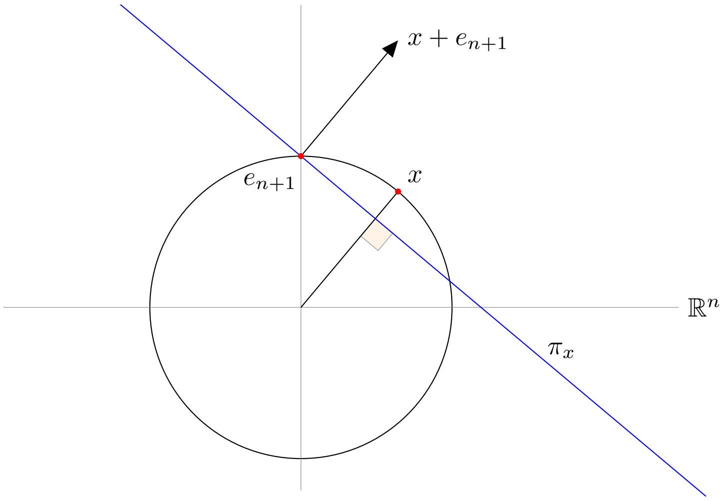


Figure 3.2: The construction of π_x

$\pi_x = \pi_{-x}$.

Now let $f_i(x)$ be the volume (measure) of the part of X_i which lies on the same side of π_x as $x + e_{n+1}$. Thus $f_i(x)$ and $f_i(-x)$ are the volumes of the bits of X_i on either side of π_x.

It can be shown[3] that each $f_i : S^n \to \mathbb{R}$ is continuous so that the map $f = (f_1, \ldots, f_n) : S^n \to \mathbb{R}^n$ is also continuous. Now Corollary 3.13 applies to give an $x_0 \in S^n$ such that $f(x_0) = f(-x_0)$. Otherwise said, $\pi_{x_0} = \pi_{-x_0}$ bisects each X_i, as required. □

[2]We really mean the same Lebesgue measure and so area when $n = 2$.
[3]But we shall not, which is why this is a sketch proof!