

Section A

1. Which of the following subsets of \mathbb{R}^3 are linear subspaces? In each case, briefly justify your answer.

(a) $U_1 := \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 = x_2^2\}$;

(b) $U_2 := \mathbb{Z}^3 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1, x_2, x_3 \in \mathbb{Z}\}$;

(c) $U_3 := \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid (x_1 - x_3)^2 + x_2^2 = 0\}$.

[4]

2. Let $U = \text{span}\{(1, 2, 1)\} \leq \mathbb{R}^3$. Which of the following equalities hold in the quotient space \mathbb{R}^3/U ? In each case, briefly justify your answer.

(a) $(1, 2, 3) + U = (4, 8, 7) + U$;

(b) $((1, 1, 2) + U) + ((2, 7, 2) + U) = (-2, -2, -1) + U$.

[4]

3. Compute the minimum polynomial and Jordan normal form of the matrix $A \in M_2(\mathbb{C})$ given by

$$A = \begin{pmatrix} -1 & 1 \\ -4 & 3 \end{pmatrix}.$$

[4]

4. With A as in question 3, compute $A^4 - 4A^3 + 6A^2 - 4A + I_2$, where I_2 is the 2×2 identity matrix.

[4]

5. For each pair (p, q) below, either give an example of a quadratic form on \mathbb{R}^5 with signature (p, q) or explain why one does not exist.
- (i) $(3, 2)$.
 - (ii) $(2, 0)$.
 - (iii) $(3, 3)$.
 - (iv) $(-2, 2)$.

[4]

Section B

6. (a) Let V be a finite-dimensional vector space over a field \mathbb{F} and let $U_1, U_2, U_3 \leq V$ be subspaces. Suppose that

$$\dim(U_1 + U_2 + U_3) = \dim U_1 + \dim U_2 + \dim U_3.$$

Show that the sum $U_1 + U_2 + U_3$ is direct.

[State any results from lectures that you use.]

[6]

- (b) Let A be given by

$$A = \begin{pmatrix} -1 & 1 & 0 \\ -3 & 3 & -1 \\ -2 & 1 & 1 \end{pmatrix}.$$

- (i) Compute the characteristic and minimum polynomials of A .
- (ii) Find the Jordan normal form of A .
- (iii) Find a Jordan basis for A .

[9]

7. (a) Let A be a square matrix with characteristic polynomial $(x-2)^4(x-1)^2$ and minimal polynomial $(x-2)^2(x-1)$. What are the possibilities for the Jordan normal form of A ? [6]
- (b) Let $t \in \mathbb{R}$ and define a quadratic form $q_t : \mathbb{R}^3 \rightarrow \mathbb{R}$ by

$$q_t(x) = x_1^2 + 2x_2^2 - x_3^2 + 2tx_1x_3 + 4x_1x_2.$$

What is the rank and signature of q_t ? (Your answer will depend on t .)

[9]