MA22020 Mock exam sol$^{\text{n}}$s

1/ (a) Not a subspace since not closed under add$^{\text{n}}$: $(1,1,0),(1,-1,0) \in U_1$ but $(2,0,0) \notin U_1$ [1]

(b) Not closed under scalar mult$^{\text{n}}$; $(1,0,0) \in U_2$ but $\frac{1}{2}(1,0,0) = (\frac{1}{2},0,0) \notin U_2$ [1]

(c) $x \in U_3$ iff $x_1 = x_3$, $x_2 = 0$

$\therefore$ $U_3 = \ker \varphi_A$ where $A = (1,0,-1)$

& $U_3 \leq \mathbb{R}^3$ [2]

2/ (a) $(4,8,7) - (1,2,3) = (3,6,4) \notin U$

so $(1,2,3) + U \neq (4,8,7) + U$ [2]

(b) $((1,1,2) + U) + ((2,7,2) + U) = (3,8,4) + U$

contd: $(3,8,4) - (-2,-2,-1) =$

$(5,10,5) = 5(1,2,1) \in U$ so

equality holds. [2]

3/ $\det(A - xI_2) = \begin{vmatrix} -1-x & 1 \\ -4 & 3-x \end{vmatrix} = -(1+x)(3-x) + 4$

$= -3 - 2x + x^2 + 4$

$= x^2 - 2x + 1 = (x-1)^2$

$A \neq I_2$ so $(x-1)^2$ is min. poly.

$\therefore$ Largest Jordan block is 2×2 so

JNF is $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

[4]

4/ Let $p = x^4 - 4x^3 + 6x^2 - 4x + 1$
$= (x-1)^4$

Then $p(A) = m_A(A)^2 = 0$ [4]

[Drop a mark if just multiply out]

5/ (i) $q(x) = x_1^2 + x_2^2 + x_3^2 - x_4^2 - x_5^2$ [1]

(ii) $q(x) = x_1^2 + x_2^2$ [1]

(iii) $3+3 = 6 > 5$ $\cancel{\exists}$ since rank $\leq$ dim [1]

(iv) $\cancel{\exists}$ sig is (p,q) with p,q dimensions

$\therefore \geq 0$. [1]

6/ (a) Let $\mathcal{B}_1 : v_1, \dots, v_k$ be basis of U_1

$\mathcal{B}_2 : v_{k+1}, \dots, v_e$ " " of U_2

$\mathcal{B}_3 : v_{e+1}, \dots, v_n$ be basis of U_3

then $v_1, \dots, v_n$ span $U_1 + U_2 + U_3$

but $\dim U_1 + U_2 + U_3 = n$

$\therefore$ $v_1, \dots, v_n$ are a basis of $U_1 + U_2 + U_3$

Lectures: $U_1 + U_2 + U_3$ direct $\Longleftrightarrow$ $\mathcal{B}_1 \mathcal{B}_2 \mathcal{B}_3$ basis of $U_1 + U_2 + U_3$

$\therefore$ $U_1 + U_2 + U_3$ direct. [6]

Made with Goodnotes

6(b)

(i) $\begin{vmatrix} -1-x & 1 & 0 \\ -3 & 3-x & -1 \\ -2 & 1 & 1-x \end{vmatrix} = -(1+x)\Big((3-x)(1-x)+1\Big) - \big(3(x-1)-2\big)$

$= -(1+x)\left(4+x^2-4x\right) - (3x-5)$

$= -x^3+4x^2-4x-4-x^2+4x-3x+5$

$= -x^3+3x^2-3x+1 = -(x-1)^3$

$\therefore$ possible min polyn $(x-1)$, $(x-1)^2$

$A \neq I_3$ so not $x-1$.

$(A-I_3)^2 = \begin{pmatrix} -2 & 1 & 0 \\ -3 & 2 & -1 \\ -2 & 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} -2 & 1 & 0 \\ -3 & 2 & -1 \\ -2 & 1 & 0 \end{pmatrix}\begin{pmatrix} -2 & 1 & 6 \\ -3 & 2 & -1 \\ -2 & 1 & 0 \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 & -1 \\ 2 & 0 & -2 \\ 1 & 0 & -1 \end{pmatrix} \neq 0 \qquad (*)$

$\therefore$ min poly $= -$char poly $= (x-1)^3$ [3]

(ii) Only eval is 1 $\therefore$ am $= 3 =$ size of largest Jordan block so

JNF $= \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ [2]

Made with Goodnotes

(iii) Jordan basis is v_1, v_2, v_3 with

$(A-I)v_3 = v_2$ $(A-I)v_2 = v_1$ $(A-I)v_1 = 0$ so need

v_3 with $(A-I)^2 v_3 \neq 0$

From (*) take $v_3 = (1,0,0)$ then

$v_2 = (-2,-3,-2)$ & $v_1 = (1,2,1)$. [4]

7(a) From char poly $A \in M_6(\mathbb{C})$

with evals 2, 1 of alg mult 4, 2

resp. $\therefore$ $\dim G_A(2) = 4$, $\dim G_A(1) = 2$

From min poly: largest Jordan block size for

eval 2 is 2 & for 1 is 1

$\therefore$ since sizes of Jordan block add to a.m.,

$J(2,2) \oplus J(2,2) \oplus J(1,1) \oplus J(1,1)$ or

$J(2,2) \oplus J(2,1) \oplus J(2,1) \oplus J(1,1) \oplus J(1,1)$

[6]

(b) Polarisation of q_t is B_{A_t} with

$$A_t = \begin{pmatrix} 1 & 2 & t \\ 2 & 2 & 0 \\ t & 0 & -1 \end{pmatrix}.$$

Diagonalise this exploiting 0 in 2,3 slot to see that e_2, e_3 is start of a diagonalising basis. Last elt is $y = v_3$ where

$$(0,1,0) A_t \underline{y} = 0 \quad \text{i.e. } (2,2,0)\underline{y} = 2y_1 + 2y_2 = 0$$

$$(0,0,1) A_t \underline{y} = 0 \quad \text{i.e. } (t,0,-1)\underline{y} = 0$$

i.e. $ty_1 = y_3 \quad \therefore \quad y = (1, -1, t) = v_3$

Now $q_t(v_3) = 1 + 2 - t^2 + 2t^2 - 4$

$$= t^2 - 1 \;\; \begin{cases} > 0 & |t| > 1 \\ 0 & |t| = 1 \\ < 0 & |t| < 1 \end{cases}$$

$$\operatorname{rank} q_t = \begin{cases} 3 & |t| \neq 1 \\ 2 & |t| = 1 \end{cases}$$

$$\operatorname{sig} q_t = \begin{cases} (2,1) & |t| > 1 \\ (1,1) & |t| = 1 \\ (1,2) & |t| < 1 \end{cases}$$

[9]