

## MA22020: Exercise sheet 6

### Warmup questions

1. Let  $B : V \times V \rightarrow \mathbb{F}$  be a symmetric bilinear form with diagonalising basis  $v_1, \dots, v_n$ . Suppose that, for some  $v_i$ ,  $1 \leq i \leq n$ , we have  $B(v_i, v_i) = 0$ . Prove that  $v_i \in \text{rad } B$ .
2. Let  $B : V \times V \rightarrow \mathbb{F}$  be a real symmetric bilinear form with diagonalising basis  $v_1, \dots, v_n$ . Show that  $B$  is positive definite if and only if  $B(v_i, v_i) > 0$ , for all  $1 \leq i \leq n$ .
3. Let  $A, B \in M_{n \times n}(\mathbb{F})$  be congruent:  $B = P^T A P$ , for some  $P \in \text{GL}(n, \mathbb{F})$ . Are the following statements true or false?
  - (a)  $\det A = \det B$ .
  - (b)  $A$  is symmetric if and only if  $B$  is symmetric.

### Rank and signature

4. Let  $B = B_A : \mathbb{R}^4 \times \mathbb{R}^4 \rightarrow \mathbb{R}$  where

$$A = \begin{pmatrix} 0 & 2 & 1 & 0 \\ 2 & 0 & 0 & 1 \\ 1 & 0 & 0 & 2 \\ 0 & 1 & 2 & 0 \end{pmatrix}.$$

Diagonalise  $B$  and hence, or otherwise, compute its signature.

5. Diagonalise the symmetric bilinear form  $B : \mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R}$  given by  $B(x, y) = x_1 y_1 + x_1 y_2 + x_2 y_1 + 2x_2 y_2 + x_2 y_3 + x_3 y_2 + x_3 y_3$ . Hence, or otherwise, compute the rank and signature of  $B$ .
6. Compute the rank and signature of the quadratic form  $Q(x) = x_1 x_2 - 4x_3 x_4$  on  $\mathbb{R}^4$ .

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## MA22020: Exercise sheet 6—Solutions

1. In this case, we have  $B(v_i, v_j) = 0$ , for all  $1 \leq j \leq n$ . So, if  $v \in V$ , write  $v = \sum_j \lambda_j v_j$  and then

$$B(v_i, v) = \sum_j \lambda_j B(v_i, v_j) = 0.$$

Otherwise said,  $v_i \in \text{rad } B$ .

2. If  $B$  is positive definite, then  $B(v, v) > 0$  for any non-zero  $v \in V$  and so, in particular, each  $B(v_i, v_i) > 0$ .

Conversely, suppose that each  $B(v_i, v_i) > 0$  and let  $v \in V$ . Write  $v = \lambda_1 v_1 + \dots + \lambda_n v_n$  and compute:

$$B(v, v) = B\left(\sum_i \lambda_i v_i, \sum_j \lambda_j v_j\right) = \sum_{i,j} \lambda_i \lambda_j B(v_i, v_j) = \sum_i \lambda_i^2 B(v_i, v_i).$$

This last is non-negative and vanishes if and only if each  $\lambda_i^2 B(v_i, v_i) = 0$ , or, equivalently,  $\lambda_i = 0$ . Thus  $B$  is positive definite.

3. (a) This is false: let  $P = \lambda I_n$ , for  $\lambda \in \mathbb{F}$ . Then  $B = \lambda^2 A$  so that  $\det B = \lambda^{2n} \det A$ .  
(b) This is true: if  $A^T = A$  then

$$B^T = (P^T A P)^T = P^T A^T P = P^T A P = B.$$

Conversely, if  $B^T = B$  we get  $P^T A^T P = P^T A P$  and multiplying by  $P^{-1}$  on the right and  $(P^T)^{-1}$  on the left gives  $A^T = A$ .

4. We need to start with  $v_1$  with  $B(v_1, v_1) \neq 0$ . Those diagonal zeros say that none of the standard basis will do so let us try  $v_1 = (1, 1, 0, 0)$  for which  $B(v_1, v_1) = 4$ .

Now seek  $v_2$  among the  $y$  with

$$0 = B(v_1, y) = \begin{pmatrix} 1 & 1 & 0 & 0 \end{pmatrix} A \mathbf{y} = \begin{pmatrix} 2 & 2 & 1 & 1 \end{pmatrix} \mathbf{y} = 2y_1 + 2y_2 + y_3 + y_4.$$

We take  $v_2 = (0, 0, 1, -1)$  with

$$B(v_2, y) = \begin{pmatrix} 0 & 0 & 1 & -1 \end{pmatrix} A \mathbf{y} = \begin{pmatrix} 1 & -1 & -2 & 2 \end{pmatrix} \mathbf{y} = y_1 - y_2 - 2y_3 + 2y_4.$$

Then  $B(v_2, v_2) = -4$  and we seek  $v_3$  among the  $y$  with  $B(v_1, y) = B(v_2, y) = 0$ , that is:

$$2y_1 + 2y_2 + y_3 + y_4 = 0$$

$$y_1 - y_2 - 2y_3 + 2y_4 = 0.$$

One solution is  $v_3 = (-3, 5, -4, 0)$  with

$$B(v_3, y) = \begin{pmatrix} -3 & 5 & -4 & 0 \end{pmatrix} A \mathbf{y} = 3 \begin{pmatrix} 2 & -2 & -1 & -1 \end{pmatrix} \mathbf{y} = 3(2y_1 - 2y_2 - y_3 - y_4).$$

Thus  $B(v_3, v_3) = -36$  and we need to find  $v_4 = y$  with  $B(v_1, y) = B(v_2, y) = B(v_3, y) = 0$ :

$$2y_1 + 2y_2 + y_3 + y_4 = 0$$

$$y_1 - y_2 - 2y_3 + 2y_4 = 0$$

$$2y_1 - 2y_2 - y_3 - y_4 = 0.$$

A solution is  $v_4 = (0, 4, -5, -3)$  with  $B(v_4, v_4) = 36$ .

We now have a diagonalising basis with  $B(v_i, v_i) = 4, -4, -36, 36$  so  $B$  has signature  $(2, 2)$  and so has rank 4.

After all this linear equation solving it is probably good to check our answer: let  $P$  have the  $v_j$  as columns and check that  $P^T A P$  is diagonal:

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ -3 & 5 & -4 & 0 \\ 0 & 4 & -5 & -3 \end{pmatrix} \begin{pmatrix} 0 & 2 & 1 & 0 \\ 2 & 0 & 0 & 1 \\ 1 & 0 & 0 & 2 \\ 0 & 1 & 2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & -3 & 0 \\ 1 & 0 & 5 & 4 \\ 0 & 1 & -4 & -5 \\ 0 & -1 & 0 & -3 \end{pmatrix} = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & -36 & 0 \\ 0 & 0 & 0 & 36 \end{pmatrix}$$

5.  $B = B_A$  where

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

Let us exploit the zero in the  $(1, 3)$  slot: note that

$$B(e_1, e_1) = B(e_3, e_3) = 1, \quad B(e_1, e_3) = 0$$

so that we just need to find  $y$  with

$$0 = B(e_1, y) = y_1 + y_2$$

$$0 = B(e_3, y) = y_2 + y_3.$$

Clearly  $y = (1, -1, 1)$  does the job with  $B(y, y) = 0$ . Thus  $e_1, e_3, y$  are a diagonalising basis with matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Either way, we see that the signature is  $(2, 0)$  and so the rank is 2.

6. The fastest way to do this is to recall that  $xy = \frac{1}{4}((x+y)^2 - (x-y)^2)$  so that

$$x_1 x_2 - 4x_3 x_4 = \frac{1}{4}(x_1 + x_2)^2 - \frac{1}{4}(x_1 - x_2)^2 - (x_3 + x_4)^2 + (x_3 - x_4)^2.$$

Moreover, the four linear functions  $x_1 \pm x_2, x_3 \pm x_4$  have linearly independent coefficients:  $(1, \pm 1, 0, 0)$  and  $(0, 0, 1, \pm 1)$ .

Now two squares appear positively and two negatively giving signature  $(2, 2)$  and so rank 4.