MA22020: Exercise sheet 5

1. Let $\lambda \in \mathbb{F}$ and define $J(\lambda, n) \in M_n(\mathbb{F})$ by

$$J(\lambda, n) = \begin{pmatrix} \lambda & 1 & 0 & \dots & 0 \\ & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & \ddots & 0 \\ & & & \ddots & 1 \\ 0 & & & & \lambda \end{pmatrix}.$$

   Set $J_n := J(0, n)$.
   Prove:
   (a) $\ker J_n^k = \operatorname{span}\{e_1, \dots, e_k\}$.
   (b) $\operatorname{im} J_n^k = \operatorname{span}\{e_1, \dots, e_{n-k}\}$.
   (c) $m_{J(\lambda,n)} = \pm\Delta_{J(\lambda,n)} = (x - \lambda)^n$.
   (d) λ is the only eigenvalue of $J(\lambda, n)$ and $E_{J(\lambda,n)}(\lambda) = \operatorname{span}\{e_1\}$, $G_{J(\lambda,n)}(\lambda) = \mathbb{F}^n$.

2. Let $v_1, \dots, v_n$ be a basis for a vector space and $\phi \in L(V)$. Show that the following are equivalent:
   (1) $\phi(v_1) = 0$ and $\phi(v_i) = v_{i-1}$, for $2 \leq i \leq n$.
   (2) $v_i = \phi^{n-i}(v_n)$ and $\phi^n(v_n) = 0$.

3. Let $\phi \in L(V)$ be a nilpotent linear operator on a finite-dimensional vector space V with Jordan normal form $J_{n_1} \oplus \dots \oplus J_{n_k}$.
   Show that

$$\#\{i \mid n_i = s\} = 2 \dim \ker \phi^s - \dim \ker \phi^{s-1} - \dim \ker \phi^{s+1}.$$

4. Let $\phi \in L(V)$ be a nilpotent linear operator on a finite-dimensional vector space V with Jordan normal form $J_{n_1} \oplus \dots \oplus J_{n_k}$.
   Use question 3 on sheet 4 to show that $m_\phi = x^s$ where $s = \max\{n_1, \dots, n_k\}$.

5. Let $\phi \in L(V)$ be a linear operator on a finite-dimensional complex vector space V with distinct eigenvalues $\lambda_1, \dots, \lambda_k$.
   Show that ϕ is diagonalisable if and only if $m_\phi = \prod_{i=1}^k (x - \lambda_i)$.

6. Let $\phi = \phi_A \in L(\mathbb{C}^3)$ where A is given by

$$\begin{pmatrix} 0 & 0 & 0 \\ 4 & 0 & 0 \\ 0 & 0 & 5 \end{pmatrix}.$$

   Find the JNF and a Jordan basis for ϕ.
   (You have studied ϕ before in question 4 of sheet 4.)

7. Let $\phi = \phi_A \in L(\mathbb{C}^3)$ where

$$\begin{pmatrix} 0 & 1 & -1 \\ -10 & -2 & 5 \\ -6 & 2 & 1 \end{pmatrix}.$$

   Find the JNF and a Jordan basis for ϕ.
   (You have studied ϕ before in question 6 of sheet 4.)

MA22020: Exercise sheet 5—Solutions

1. Note that $\phi_{J_n}(x) = (x_2, \dots, x_n, 0)$ so that $\phi^k_{J_n}(x) = (x_{k+1}, \dots, x_n, 0, \dots, 0)$, $k < n$ and $\phi^n_{J_n} = 0$.
   (a) It is clear from the above that $\ker J_n^k = \{x \in \mathbb{F}^n \mid x_{k+1} = \cdots = x_n = 0\} = \operatorname{span}\{e_1, \dots, e_k\}$.
   (b) Similarly, $\operatorname{im} J_n^k = \{y \in \mathbb{F}^n \mid y_{n-k+1} = \cdots = y_n = 0\} = \operatorname{span}\{e_1, \dots, e_{n-k}\}$.
   (c) $J(\lambda, n)$ is upper triangular so that $\Delta_{J(\lambda,n)} = (\lambda - x)^n$. Therefore $m_{J(\lambda,n)} = (x-\lambda)^s$, for some $s \leq n$. However $(J(\lambda, n) - \lambda I_n)^k = J_n^k \neq 0$, for $k < n$, so that $m_{J(\lambda,n)} = (x - \lambda)^n$.
   (d) Finally, it is clear that λ is the only eigenvalue and the eigenspace is $\ker(J(\lambda, n) - \lambda I_n) = \ker J_n = \operatorname{span}\{e_1\}$ by part (a). Similarly, $G_{J(\lambda,n)}(\lambda) = \ker J_n^n = \mathbb{F}^n$.

2. Assume (1). Then $v_{n-1} = \phi(v_n)$ and induction gives $v_i = \phi^{n-i}(v_n)$. In particular, $v_1 = \phi^{n-1}(v_n)$ so that $\phi(v_1)$ gives $\phi^n(v_n) = 0$. This establishes (2).
   Assume (2). Then $0 = \phi^n(v_n) = \phi(\phi^{n-1}(v_n)) = \phi(v_1)$. Moreover $\phi(v_i) = \phi(\phi^{n-i}(v_n)) = \phi^{n-(i-1)}(v_n) = v_{i-1}$, for $2 \leq i \leq n$. Thus we have (1).

3. From lectures, we know that, for $s \geq 1$,
   $$\#\{i \mid n_i \geq s\} = \dim\ker\phi^s - \dim\ker\phi^{s-1}.$$
   Now
   $$\begin{aligned}\#\{i \mid n_i = s\} &= \#\{i \mid n_i \geq s\} - \#\{i \mid n_i \geq s+1\} \\ &= \dim\ker\phi^s - \dim\ker\phi^{s-1} - (\dim\ker\phi^{s+1} - \dim\ker\phi^s) \\ &= 2\dim\ker\phi^s - \dim\ker\phi^{s-1} - \dim\ker\phi^{s+1}.\end{aligned}$$

4. We follow the hint which tells us that when $A = A_1 \oplus \cdots \oplus A_k$, m_A is the smallest monic polynomial divided by each m_{A_i}.
   In the case at hand, with $A = J_{n_1} \oplus \cdots \oplus J_{n_k}$, we know from question 1 that $m_{J_{n_i}} = x^{n_i}$. Thus m_A is the monic polynomial of smallest degree divided by each x^{n_i} which is x^s, for $s = \max\{n_1, \dots, n_k\}$.

5. Let $p = \prod_{i=1}^k (x - \lambda_i) \in \mathbb{C}[x]$.
   If ϕ is diagonalisable then $p(\phi) = 0$ since $\phi - \lambda_i \operatorname{id}_V = 0$ on $E_\phi(\lambda_i)$ and V is the direct sum of these eigenspaces.
   Conversely, if $m_\phi = \prod_{i=1}^k (x - \lambda_i)$, a result from lectures tells us that all Jordan blocks in the Jordan normal form of ϕ have size 1. Otherwise said, the JNF is diagonal and the Jordan basis is an eigenbasis. So ϕ is diagonalisable.

6. From sheet 4, we have that $m_\phi = x^2(x-5)$ so that the JNF is $J(0, 2) \oplus J(5, 1)$. A Jordan basis for $G_\phi(5)$ is any non-zero eigenvector with eigenvalue 5. We know from sheet 4 that $(0, 0, 1)$ is such a vector.
   For $G_\phi(0)$, a Jordan basis is $\phi(v), v$ with $\phi^2(v) = 0$. As usual, work backwards from $w \in \ker\phi$: take $w = (0, 1, 0)$ and solve
   $$A\mathbf{v} = \begin{pmatrix} 0 & 0 & 0 \\ 4 & 0 & 0 \\ 0 & 0 & 5 \end{pmatrix} \mathbf{v} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$
   to get $v = (1/4, 0, 0)$, for example.
   Thus, a Jordan basis is given by $(0, 1, 0), (1/4, 0, 0), (0, 0, 1)$.

7. From question 6 of sheet 4, we know that $m_\phi = (x-3)(x+2)^2$ so that the JNF of ϕ must be $J(3, 1) \oplus J(-2, 2)$. A Jordan basis of $G_\phi(3) = E_\phi(3)$ is an arbitrary basis and one is given by $(0, 1, 1)$ as we found out in question 3 of sheet 4.

For $G_\phi(-2)$, we want $(\phi + 2\,\mathrm{id}_{\mathbb{C}^3})(v), v$ with $(\phi + 2\,\mathrm{id}_{\mathbb{C}^3})^2(v) = 0$ so work backwards from an eigenvector w with eigenvalue -2 and then solve $(A + 2I_3)\mathbf{v} = \mathbf{w}$ to get v. We know from sheet 4 that we can take $w = (1, 0, 2)$ and then

$$(A + 2I_3)\mathbf{v} = \begin{pmatrix} 2 & 1 & -1 \\ -10 & 0 & 5 \\ -6 & 2 & 3 \end{pmatrix} \mathbf{v} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$

is clearly solved by $(0, 1, 0)$. Our Jordan basis is therefore $(0, 1, 1), (1, 0, 2), (0, 1, 0)$.