MA22020: Exercise sheet 4

Warmup questions

1. Write down matrices $A \in M_n(\mathbb{R})$ of the following forms:
   (a) $A_1 \oplus A_2 \oplus A_3$ with each $A_i \in M_2(\mathbb{R})$.
   (b) $A_1 \oplus \cdots \oplus A_5$ with each $A_i \in M_1(\mathbb{R})$.
   (c) $A \in M_3(\mathbb{R})$ such that A is not of the form $A_1 \oplus \cdots \oplus A_k$ with $k > 1$.
2. Let $V_1, \dots, V_k \leq V$ and $\phi_i \in L(V_i)$, $1 \leq i \leq k$. Suppose that $V = V_1 \oplus \cdots \oplus V_k$ and set $\phi = \phi_1 \oplus \cdots \oplus \phi_k$.
   (a) If $U_i \leq V_i$, $1 \leq i \leq k$, show that the sum $U_1 + \cdots + U_k$ is direct.
   (b) Prove that $\operatorname{im} \phi = \operatorname{im} \phi_1 \oplus \cdots \oplus \operatorname{im} \phi_k$.
3. In the situation of Question 2, prove:
   (a) m_{ϕ_i} divides m_ϕ, for each $1 \leq i \leq k$.
   (b) If each m_{ϕ_i} divides $p \in \mathbb{F}[x]$, then $p(\phi) = 0$.

   Thus m_ϕ is the monic polynomial of smallest degree divided by each m_{ϕ_i}. Otherwise said, m_ϕ is the *least common multiple* of $m_{\phi_1}, \dots, m_{\phi_k}$.
4. Let $\phi = \phi_A \in L(\mathbb{C}^3)$ where A is given by

$$\begin{pmatrix} 0 & 0 & 0 \\ 4 & 0 & 0 \\ 0 & 0 & 5 \end{pmatrix}.$$

   (a) Compute the characteristic and minimum polynomials of ϕ.
   (b) Find bases for the eigenspaces and generalised eigenspaces of ϕ.

Homework questions

5. Let $\phi \in L(V)$ be a linear operator on a vector space V.
   Prove that $\operatorname{im} \phi^k \geq \operatorname{im} \phi^{k+1}$, for all $k \in \mathbb{N}$. Moreover, if $\operatorname{im} \phi^k = \operatorname{im} \phi^{k+1}$ then $\operatorname{im} \phi^k = \operatorname{im} \phi^{k+n}$, for all $n \in \mathbb{N}$.
6. Let $\phi = \phi_A \in L(\mathbb{C}^3)$ where A is given by

$$\begin{pmatrix} 0 & 1 & -1 \\ -10 & -2 & 5 \\ -6 & 2 & 1 \end{pmatrix}.$$

   (a) Compute the characteristic and minimum polynomials of ϕ.
   (b) Find bases for the eigenspaces and generalised eigenspaces of ϕ.

Please hand in at 4W level 1 by NOON on Thursday 27th November

MA22020: Exercise sheet 4—Solutions

1. There are a gazillion possibilities.

   (*a*)
$$\begin{pmatrix}1&2&0&0&0&0\\2&4&0&0&0&0\\0&0&5&6&0&0\\0&0&7&8&0&0\\0&0&0&0&9&0\\0&0&0&0&1&2\end{pmatrix}=\begin{pmatrix}1&2\\3&4\end{pmatrix}\oplus\begin{pmatrix}5&6\\7&8\end{pmatrix}\oplus\begin{pmatrix}9&0\\1&2\end{pmatrix}.$$

   (*b*) Any 5×5 diagonal matrix will do:
$$\begin{pmatrix}\lambda_1&0&0&0&0\\0&\lambda_2&0&0&0\\0&0&\lambda_3&0&0\\0&0&0&\lambda_4&0\\0&0&0&0&\lambda_5\end{pmatrix}=(\lambda_1)\oplus\cdots\oplus(\lambda_5).$$

   (*c*) Any block matrix with more than one block will have zeros so
$$A=\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}$$
   cannot be written $A_1\oplus\cdots\oplus A_k$ with $k>1$.

2. (*a*) Let $u\in U_1+\cdots+U_k$ so we can write $u=u_1+\cdots+u_k$, with $u_i\in U_i\leq V_i$. However, since the sum of the V_i is direct, there is only one way to write u as a sum of elements of the V_i and so, in particular, as a sum of elements of the U_i. Thus the sum of the U_i is direct.

   (*b*) Let $v\in\operatorname{im}\phi$ so that $v=\phi(w)$, for some $w\in V$. Then, writing $w=w_1+\cdots+w_k$ with each $w_i\in V_i$, we have
$$v=\phi(w)=\phi_1(w_1)+\cdots+\phi_k(w_k)\in\operatorname{im}\phi_1\oplus\cdots\oplus\operatorname{im}\phi_k.$$
   Thus $\operatorname{im}\phi\leq\operatorname{im}\phi_1\oplus\cdots\oplus\operatorname{im}\phi_k$.
   For the converse, let $v\in\operatorname{im}\phi_1\oplus\cdots\oplus\operatorname{im}\phi_k$ so that $v=\phi_1(w_1)+\cdots+\phi_k(w_k)$ with $w_i\in V_i$, $1\leq i\leq k$. Since each $\phi_i=\phi_{|V_i}$, this reads
$$v=\phi(w_1)+\cdots+\phi(w_k)=\phi(w_1+\cdots+w_k)\in\operatorname{im}\phi.$$
   Thus $\operatorname{im}\phi_1\oplus\cdots\oplus\operatorname{im}\phi_k\leq\operatorname{im}\phi$ and we are done.

3. (*a*) We have that $m_\phi(\phi)=0$ so that $0=m_\phi(\phi)_{|V_i}=m_\phi(\phi_i)$. It follows that m_{ϕ_i} divides m_ϕ.

   (*b*) Since m_{ϕ_i} divides p, $p(\phi_i)=0$ for each i. But then
$$p(\phi)=p(\phi_1)\oplus\cdots\oplus p(\phi_k)=0$$
   so that m_ϕ divides p.

4. (*a*) Since A is lower triangular, we immediately see that $\Delta_\phi=\Delta_A=x^2(x-5)$. So the only possibilities for $m_\phi=x(x-5)$ and $x^2(x-5)$. However
$$A-5I_3=\begin{pmatrix}-5&0&0\\4&-5&0\\0&0&0\end{pmatrix}$$

so that

$$A(A - 5I_3) = \begin{pmatrix} 0 & 0 & 0 \\ -20 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \neq 0.$$

We conclude that $m_\phi = x^2(x-5)$.
Alternatively, A is block diagonal:

$$A = \begin{pmatrix} 0 & 0 \\ 4 & 0 \end{pmatrix} \oplus (5)$$

and the summands clearly have minimum polynomials x^2 and $x-5$ respectively. It follows question 3 that $m_\phi = x^2(x-5)$.

(*b*) We have $E_\phi(5) = G_\phi(5) = \operatorname{span}\{(0,0,1)\}$, $E_\phi(0) = \ker A = \operatorname{span}\{(0,1,0)\}$ and finally $G_\phi(0) = \ker A^2 = \operatorname{span}\{(1,0,0),(0,1,0)\}$ since

$$A^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 25 \end{pmatrix}.$$

5. Let $v \in \operatorname{im}\phi^{k+1}$ so that $v = \phi^{k+1}(w)$, for some $w \in V$. Then $v = \phi^k(\phi(w)) \in \operatorname{im}\phi^k$. Thus $\operatorname{im}\phi^k \geq \operatorname{im}\phi^{k+1}$.
Suppose now that $\operatorname{im}\phi^k = \operatorname{im}\phi^{k+1}$. We prove that $\operatorname{im}\phi^k = \operatorname{im}\phi^{k+n}$ by induction on n. We are given that this holds for $n = 1$ so we suppose this holds for some n ($\operatorname{im}\phi^k = \operatorname{im}\phi^{k+n}$) and prove it then holds for $n+1$. Thus, let $v \in \operatorname{im}\phi^k = \operatorname{im}\phi^{k+1}$ so that $v = \phi(\phi^k(w))$, for some $w \in V$. Then $\phi^k(w) \in \operatorname{im}\phi^k = \operatorname{im}\phi^{k+n}$, by the induction hypothesis, so that $\phi^k(w) = \phi^{k+n}(u)$, some $u \in V$, whence $v = \phi(\phi^{k+n}(u)) = \phi^{k+n+1}(u) \in \operatorname{im}\phi^{k+n+1}$. We conclude that $\operatorname{im}\phi^k \leq \operatorname{im}\phi^{k+n+1}$. The converse inclusion always holds so we have equality. Induction now bakes the cake.

6. (*a*) We compute the characteristic polynomial: $\Delta_\phi = \Delta_A = -x^3 - x^2 + 8x + 12 = (3-x)(x+2)^2$. Consequently, m_ϕ is either $(x-3)(x+2)^2$ or $(x-3)(x+2)$. We try the latter:

$$A - 3I_3 = \begin{pmatrix} -3 & 1 & -1 \\ -10 & -5 & 5 \\ -6 & 2 & -2 \end{pmatrix} \qquad A + 2I_3 = \begin{pmatrix} 2 & 1 & -1 \\ -10 & 0 & 5 \\ -6 & 2 & 3 \end{pmatrix}$$

so that

$$(A - 3I_3)(A + 2I_3) = \begin{pmatrix} -10 & -5 & 5 \\ 0 & 0 & 0 \\ -20 & -10 & 10 \end{pmatrix} \neq 0.$$

Thus $m_\phi = m_A = (x-3)(x+2)^2$.

(*b*) We deduce that $G_\phi(3) = E_\phi(3) = \ker(A - 3I_3)$ while $E_\phi(-2) = \ker(A + 2I_3)$ and $G_\phi(-2) = \ker(A + 2I_3)^2$. We compute these: an eigenvector x with eigenvalue 3 solves

$$\begin{aligned} -3x_1 + x_3 - x_3 &= 0 \\ -2x_1 - x_2 + x_3 &= 0 \end{aligned}$$

which rapidly yields $x_1 = 0$ and $x_2 = x_3$. Thus the 3-eigenspace is spanned by $(0,1,1)$.
An eigenvector x with eigenvalue 2 solves

$$\begin{aligned} 2x_1 + x_2 - x_3 &= 0 \\ -2x_1 + 0x_2 + x_3 &= 0 \end{aligned}$$

giving $x_2 = 0$ and $2x_1 = x_3$ so the eigenspace is spanned by $(1,0,2)$.

Finally,

$$(A+2I_3)^2 = \begin{pmatrix} 0 & 0 & 0 \\ -50 & 0 & 25 \\ -50 & 0 & 25 \end{pmatrix}$$

with kernel spanned by $(1,0,2)$ and $(0,1,0)$.
To summarise:

$$\begin{aligned} E_\phi(3) = G_\phi(3) &= \operatorname{span}\{(0,1,1)\} \\ E_\phi(-2) &= \operatorname{span}\{(1,0,2)\} \\ G_\phi(-2) &= \operatorname{span}\{(1,0,2),(0,1,0)\}. \end{aligned}$$