MA22020: Exercise sheet 4

Warmup questions

1. Write down matrices $A \in M_n(\mathbb{R})$ of the following forms:
   (*a*) $A_1 \oplus A_2 \oplus A_3$ with each $A_i \in M_2(\mathbb{R})$.
   (*b*) $A_1 \oplus \cdots \oplus A_5$ with each $A_i \in M_1(\mathbb{R})$.
   (*c*) $A \in M_3(\mathbb{R})$ such that A is not of the form $A_1 \oplus \cdots \oplus A_k$ with $k > 1$.

2. Let $V_1, \dots, V_k \leq V$ and $\phi_i \in L(V_i)$, $1 \leq i \leq k$. Suppose that $V = V_1 \oplus \cdots \oplus V_k$ and set $\phi = \phi_1 \oplus \cdots \oplus \phi_k$.
   (*a*) If $U_i \leq V_i$, $1 \leq i \leq k$, show that the sum $U_1 + \cdots + U_k$ is direct.
   (*b*) Prove that $\operatorname{im} \phi = \operatorname{im} \phi_1 \oplus \cdots \oplus \operatorname{im} \phi_k$.

3. In the situation of Question 2, prove:
   (*a*) m_{ϕ_i} divides m_ϕ, for each $1 \leq i \leq k$.
   (*b*) If each m_{ϕ_i} divides $p \in \mathbb{F}[x]$, then $p(\phi) = 0$.

   Thus m_ϕ is the monic polynomial of smallest degree divided by each m_{ϕ_i}. Otherwise said, m_ϕ is the **least common multiple** of $m_{\phi_1}, \dots, m_{\phi_k}$.

4. Let $\phi = \phi_A \in L(\mathbb{C}^3)$ where A is given by

$$\begin{pmatrix} 0 & 0 & 0 \\ 4 & 0 & 0 \\ 0 & 0 & 5 \end{pmatrix}.$$

   (*a*) Compute the characteristic and minimum polynomials of ϕ.
   (*b*) Find bases for the eigenspaces and generalised eigenspaces of ϕ.

Homework questions

5. Let $\phi \in L(V)$ be a linear operator on a vector space V.
   Prove that $\operatorname{im} \phi^k \geq \operatorname{im} \phi^{k+1}$, for all $k \in \mathbb{N}$. Moreover, if $\operatorname{im} \phi^k = \operatorname{im} \phi^{k+1}$ then $\operatorname{im} \phi^k = \operatorname{im} \phi^{k+n}$, for all $n \in \mathbb{N}$.

6. Let $\phi = \phi_A \in L(\mathbb{C}^3)$ where A is given by

$$\begin{pmatrix} 0 & 1 & -1 \\ -10 & -2 & 5 \\ -6 & 2 & 1 \end{pmatrix}.$$

   (*a*) Compute the characteristic and minimum polynomials of ϕ.

(*b*) Find bases for the eigenspaces and generalised eigenspaces of ϕ.

Please hand in at 4W level 1 by NOON on Thursday 28th November