MA22020: Exercise sheet 3

Warmup questions

1. Let $p, q \in \mathbb{R}[x]$ be given by $p = x^2 - 2x - 3$, $q = x^3 - 2x^2 + 2x - 5$.
   Let $A \in M_2(\mathbb{R})$ and $B \in M_3(\mathbb{R})$ be given by

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 2 & 1 \\ -2 & 0 & 1 \\ 2 & 1 & 1 \end{pmatrix}.$$

   Compute $p(A), p(B), q(A), q(B)$.

2. Compute the characteristic polynomials of A and B, from question 1.
   What do you notice?

3. Let $\mathbb{F} = \mathbb{Z}_2$, the field of two elements and let $p = x^2 + x \in \mathbb{F}[x]$.
   Show that $p(t) = 0$, for all $t \in \mathbb{F}$.

4. Let $\phi \in L(V)$ be an operator on a finite-dimensional vector space over $\mathbb{F}$.
   Show that ϕ is invertible if and only if m_ϕ has non-zero constant term.

Homework questions

5. Compute the minimum polynomial of $A \in M_5(\mathbb{R})$ given by

$$\begin{pmatrix} 0 & 0 & 0 & 0 & -3 \\ 1 & 0 & 0 & 0 & 6 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

6. Compute the characteristic and minimum polynomials of

$$A = \begin{pmatrix} 1 & -5 & -7 \\ 1 & 4 & 2 \\ 0 & 1 & 4 \end{pmatrix}.$$

Please hand in at 4W level 1 by NOON on Thursday 14th November

MA22020: Exercise sheet 3—Solutions

1. We just compute:

$$A^2 = \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix}, \qquad A^3 = \begin{pmatrix} 13 & 14 \\ 14 & 13 \end{pmatrix}$$

so that

$$p(A) = A^2 - 2A - 3I_2 = \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix} - 2\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} - 3\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$q(A) = A^3 - 2A^2 + 2A - 5I_3 = \begin{pmatrix} 13 & 14 \\ 14 & 13 \end{pmatrix} - 2\begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix} + 2\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} - 5\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 10 \\ 10 & 0 \end{pmatrix}.$$

Similarly,

$$p(B) = \begin{pmatrix} -6 & -1 & 2 \\ 4 & -6 & -3 \\ -2 & 3 & -1 \end{pmatrix},$$

$$q(B) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

2. Again, we just compute:

$$\Delta_A = \begin{vmatrix} 1-x & 2 \\ 2 & 1-x \end{vmatrix} = (1-x)^2 - 4 = x^2 - 2x - 3.$$

Similarly,

$$\Delta_B = \begin{vmatrix} 1-x & 2 & 1 \\ -2 & -x & 1 \\ 2 & 1 & 1-x \end{vmatrix} = (1-x)\big(x(x-1)-1\big) - 2\big(2(x-1)-2\big) + (-2+2x)$$

$$= (-x^3 + 2x^2 - 1) - 4x + 8 + 2x - 2 = -x^3 + 2x^2 - 2x + 5.$$

We notice that, with p, q as in question 1, $p = \Delta_A$ and $q = -\Delta_B$ and so, again from question 1,

$$\Delta_A(A) = \Delta_B(B) = 0.$$

This is the Cayley–Hamilton theorem in action.

3. We recall that $\mathbb{Z}_2 = \{\mathbf{0}, \mathbf{1}\}$ with addition and multiplication given by

$$\mathbf{0} = \mathbf{0} + \mathbf{0} = \mathbf{1} + \mathbf{1} \qquad \mathbf{1} = \mathbf{0} + \mathbf{1} = \mathbf{1} + \mathbf{0}$$

$$\mathbf{0} = \mathbf{00} = \mathbf{01} = \mathbf{10} \qquad \mathbf{1} = \mathbf{11}.$$

We immediately conclude that $\mathbf{1}^2 + \mathbf{1} = \mathbf{0} = \mathbf{0}^2 + \mathbf{0}$ so that $p(t) = \mathbf{0}$, for both $t \in \mathbb{F}$.

4. ϕ is invertible if and only if ϕ is injective if and only if zero is not an eigenvalue if and only if (thanks to the corollary to the Cayley–Hamilton theorem) zero is not a root of m_ϕ if and only if m_ϕ has non-zero constant term.

5. Let us compute the first few powers of A:

$$A^2 = \begin{pmatrix} 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 6 & -3 \\ 1 & 0 & 0 & 0 & 6 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \quad A^3 = \begin{pmatrix} 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 6 & -3 & 0 \\ 0 & 0 & 0 & 6 & -3 \\ 1 & 0 & 0 & 0 & 6 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \quad A^4 = \begin{pmatrix} 0 & -3 & 0 & 0 & 0 \\ 0 & 6 & -3 & 0 & 0 \\ 0 & 0 & 6 & -3 & 0 \\ 0 & 0 & 0 & 6 & -3 \\ 1 & 0 & 0 & 0 & 6 \end{pmatrix}$$

$$A^5 = \begin{pmatrix} -3 & 0 & 0 & 0 & -18 \\ 6 & -3 & 0 & 0 & 36 \\ 0 & 6 & -3 & 0 & 0 \\ 0 & 0 & 6 & -3 & 0 \\ 0 & 0 & 0 & 6 & -3 \end{pmatrix}$$

Stare at the top row to see that there can be no monic polynomial $p = a_0 + \cdots + x^k$ with $k \leq 4$ with $p(A) = 0$: the -3 on the top row of the leading term would give $a_0 0 + \cdots + a_{k-1} 0 - 3 = 0$. On the other hand, we readily see that $A^5 - 6A + 3I_5 = 0$ so that $m_A = x^5 - 6x + 3$.

6. We compute the characteristic polynomial of A to be

$$\Delta_A = -x^3 + 9x^2 - 27x + 27 = -(x-3)^3.$$

We learn from the Cayley–Hamilton theorem that $m_A = (x-3)^k$, for some k with $k \leq 1 \leq 3$. Clearly $k = 1$ is out, since A is not diagonal, so we try $k = 2$:

$$(A - 3I)^2 = \begin{pmatrix} -2 & -5 & -7 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} -2 & -5 & -7 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & -2 & -3 \\ -1 & -2 & -3 \\ 1 & 2 & 3 \end{pmatrix},$$

which is non-zero. This means we must have $m_A = (x-3)^3$.