

MA22020: Exercise sheet 2

Warmup questions

1. Let $U_1, U_2, U_3 \leq \mathbb{R}^3$ be the 1-dimensional subspaces spanned by $(1, 2, 0)$, $(1, 1, 1)$ and $(2, 3, 1)$ respectively.

Which of the following sums are direct?

- (a) $U_i + U_j$, for $1 \leq i < j \leq 3$.
(b) $U_1 + U_2 + U_3$.
2. Let $V_i \leq V$, for $1 \leq i \leq k$. Prove the converse of Corollary 2.8: if

$$\dim V_1 + \cdots + V_k = \dim V_1 + \cdots + \dim V_k$$

then the sum $V_1 + \cdots + V_k$ is direct.

3. Let $U \leq V$. Show that congruence modulo U is an equivalence relation.
4. Let $U = \text{span}\{(1, -1, 0), (0, 1, -1)\} \leq \mathbb{R}^3$. Determine which, if any, of the following cosets are equal:

$$(1, 2, 3) + U, \quad (3, 3, 0) + U, \quad (1, 1, 1) + U.$$

5. Let $U \leq V$ and $q : V \rightarrow V/U$ the quotient map. Let W be a complement to U . Show that $q|_W : W \rightarrow V/U$ is an isomorphism.

Homework

6. Let V be a vector space. A linear map $\pi : V \rightarrow V$ is called a *projection* if $\pi \circ \pi = \pi$. In this case, prove that $\ker \pi \cap \text{im } \pi = \{0\}$ and deduce that $V = \ker \pi \oplus \text{im } \pi$.
7. Let $U, W \leq V$. Define a linear map $\phi : U \rightarrow (U + W)/W$ by $\phi(u) = u + W$.
- (a) Use the first isomorphism theorem, applied to ϕ , to prove the second isomorphism theorem:

$$U/(U \cap W) \cong (U + W)/W.$$

- (b) Deduce that, when V is finite-dimensional,

$$\dim(U + W) = \dim U + \dim W - \dim(U \cap W).$$

Please hand in at 4W level 1 by NOON on Thursday 31st October 2024