MA22020: Exercise sheet 2

Warmup questions

1. Let $U_1, U_2, U_3 \leq \mathbb{R}^3$ be the 1-dimensional subspaces spanned by $(1,2,0)$, $(1,1,1)$ and $(2,3,1)$ respectively.
   Which of the following sums are direct?
   (a) $U_i + U_j$, for $1 \leq i < j \leq 3$.
   (b) $U_1 + U_2 + U_3$.

2. Let $V_i \leq V$, for $1 \leq i \leq k$. Prove the converse of Corollary 2.8: if

$$\dim V_1 + \cdots + V_k = \dim V_1 + \cdots + \dim V_k$$

   then the sum $V_1 + \cdots + V_k$ is direct.

3. Let $U \leq V$. Show that congruence modulo U is an equivalence relation.

4. Let $U = \operatorname{span}\{(1,-1,0),(0,1,-1)\} \leq \mathbb{R}^3$. Determine which, if any, of the following cosets are equal:

$$(1,2,3) + U, \qquad (3,3,0) + U, \qquad (1,1,1) + U.$$

5. Let $U \leq V$ and $q : V \to V/U$ the quotient map. Let W be a complement to U. Show that $q_{|W} : W \to V/U$ is an isomorphism.

Homework

6. Let V be a vector space. A linear map $\pi : V \to V$ is called a *projection* if $\pi \circ \pi = \pi$. In this case, prove that $\ker \pi \cap \operatorname{im} \pi = \{0\}$ and deduce that $V = \ker \pi \oplus \operatorname{im} \pi$.

7. Let $U, W \leq V$. Define a linear map $\phi : U \to (U+W)/W$ by $\phi(u) = u + W$.
   (*a*) Use the first isomorphism theorem, applied to ϕ, to prove the second isomorphism theorem:

$$U/(U \cap W) \cong (U+W)/W.$$

   (*b*) Deduce that, when V is finite-dimensional,

$$\dim(U+W) = \dim U + \dim W - \dim(U \cap W).$$

Please hand in at 4W level 1 by NOON on Thursday 30th October 2025

MA22020: Exercise sheet 2—Solutions

1. (*a*) All these sums are direct as each $U_i \cap U_j = \{0\}$.

   (*b*) Note that $(2,3,1) = (1,2,0) + (1,1,1)$ and so can be written in two different ways as a sum $u_1 + u_2 + u_3$, with each $u_i \in U_i$:

$$(1,2,0) + (1,1,1) + (0,0,0)$$
$$(0,0,0) + (0,0,0) + (2,3,1).$$

   Thus $U_1 + U_2 + U_3$ is not a direct sum.

   This shows us that $U_i \cap U_j = \{0\}$, $i \neq j$, is not enough to force $U_1 + U_2 + U_3$ to be direct.

2. Let $\mathcal{B}_i$ be a basis for V_i and let $\mathcal{B} = \mathcal{B}_1, \dots, \mathcal{B}_k$ be the concatenation of these. By Corollary 2.7, it suffices to see that $\mathcal{B}$ is a basis for $V_1 + \cdots + V_k$. However, $\mathcal{B}$ clearly spans and, by hypothesis,
$$|\mathcal{B}| = \dim V_1 + \cdots + \dim V_k = \dim V_1 + \cdots + V_k.$$
   We know from last year that, for any vector space W, a list of $\dim W$ vectors that span is a basis and so we are done.

3. **Reflexive** $v - v = 0 \in U$ so $v \equiv v \mod U$.

   Symmetric If $v \equiv w \mod U$ then $v - w \in U$ so that $w - v = -(v - w) \in U$ and $w \equiv v \mod U$.

   Transitive If $v \equiv w \mod U$ and $w \equiv u \mod U$, then $v - w, w - u \in U$ whence $v - u = (v - w) + (w - u) \in U$ and so $v \equiv u \mod U$.

4. There are two ways to proceed. The first is to work straight from the definitions: we know that $v_1 + U = v_2 + U$ if and only if $v_1 - v_2 \in U$, that is, $v_1 - v_2$ is a linear combination of $(1,-1,0)$ and $(0,1,-1)$. Now,
$$(3,3,0) - (1,2,3) = (2,1,-3) = 2(1,-1,0) + 3(0,1,-1) \in U$$
   so that $(3,3,0) + U = (1,2,3) + U$. On the other hand, trying to solve
$$(3,3,0) - (1,1,1) = (2,2,-1) = \lambda(1,-1,0) + \mu(0,1,-1),$$
   for $\lambda, \mu \in \mathbb{R}$, leads to inconsistent equations:
$$\lambda = 2; \qquad \mu = -1; \qquad \mu - \lambda = 2.$$
   Thus $(2,2,-1) \notin U$ and $(3,3,0) + U \neq (1,1,1) + U$.

   A slicker approach is to observe that $U = \ker\phi$ where $\phi : \mathbb{R}^3 \to \mathbb{R}$ is the linear map $\phi(x_1, x_2, x_3) = x_1 + x_2 + x_3$ (indeed, $U \leq \ker\phi$ and $\dim\ker\phi = 2$ by rank-nullity). Now $v_1 + U = v_2 + U$ if and only if $\phi(v_1) = \phi(v_2)$ and we simply compute:
$$\phi(1,2,3) = \phi(3,3,0) = 6, \qquad \phi(1,1,1) = 3$$
   to learn that
$$(1,2,3) + U = (3,3,0) + U \neq (1,1,1) + U.$$

5. Let $v \in V$. Since $V = U + W$, we write $v = u + w$ with $u \in U$ and $w \in W$. Then, since $\ker q = U$, $q(v) = q(u + w) = q(w)$ so that $\operatorname{im} q_{|W} = \operatorname{im} q = V/U$. Thus $q_{|W}$ surjects.

   Further, $\ker q_{|W} = \ker q \cap W = U \cap W = \{0\}$ since $\ker q = U$. Thus $q_{|W}$ has trivial kernel and so injects.

6. Let $v \in \ker\pi \cap \operatorname{im}\pi$. Then there is $w \in V$ such that $v = \pi(w)$ since $v \in \operatorname{im}\pi$. But $v \in \ker\pi$ also so that
$$0 = \pi(v) = \pi(\pi(w)) = \pi(w) = v.$$

Thus $\ker \pi \cap \operatorname{im} \pi = \{0\}$ so it remains to show that $V = \ker \pi + \operatorname{im} \pi$. For this, write $v = (v - \pi(v)) + \pi(v)$. The second summand is certainly in $\operatorname{im} \pi$ while

$$\pi(v - \pi(v)) = \pi(v) - \pi(\pi(v)) = \pi(v) - \pi(v) = 0$$

so the first is in $\ker \pi$ and we are done.

7. (a) Let $q : U + W \to (U + W)/W$ be the quotient map. Then ϕ is simply the restriction $q_{|U}$ of q to U and so is linear. Moreover, $\ker \phi = U \cap \ker q = U \cap W$. Finally, if $q(u + w) \in (U + W)/W$, then, since $q(w) = 0$,

$$q(u + w) = q(u) + q(w) = q(u) = \phi(u)$$

so that ϕ is onto. The first isomorphism theorem now reads

$$U/(U \cap W) = U/\ker \phi \cong \operatorname{im} \phi = (U + W)/W.$$

(b) When V is finite-dimensional, we have

$$\dim U - \dim(U \cap W) = \dim U/(U \cap W) = \dim(U + W)/W = \dim(U + W) - \dim W$$

and rearranging this gives the result.