MA22020: Exercise sheet 2

Warmup questions

1. Let $U_1, U_2, U_3 \leq \mathbb{R}^3$ be the 1-dimensional subspaces spanned by $(1,2,0)$, $(1,1,1)$ and $(2,3,1)$ respectively.
   Which of the following sums are direct?
   (a) $U_i + U_j$, for $1 \leq i < j \leq 3$.
   (b) $U_1 + U_2 + U_3$.

2. Let $V_i \leq V$, for $1 \leq i \leq k$. Prove the converse of Corollary 2.8: if

$$\dim V_1 + \cdots + V_k = \dim V_1 + \cdots + \dim V_k$$

   then the sum $V_1 + \cdots + V_k$ is direct.

3. Let $U \leq V$. Show that congruence modulo U is an equivalence relation.

4. Let $U = \operatorname{span}\{(1,-1,0),(0,1,-1)\} \leq \mathbb{R}^3$. Determine which, if any, of the following cosets are equal:

$$(1,2,3)+U, \qquad (3,3,0)+U, \qquad (1,1,1)+U.$$

5. Let $U \leq V$ and $q : V \to V/U$ the quotient map. Let W be a complement to U.
   Show that $q_{|W} : W \to V/U$ is an isomorphism.

Homework

6. Let V be a vector space. A linear map $\pi : V \to V$ is called a **projection** if $\pi \circ \pi = \pi$.
   In this case, prove that $\ker \pi \cap \operatorname{im} \pi = \{0\}$ and deduce that $V = \ker \pi \oplus \operatorname{im} \pi$.

7. Let $U, W \leq V$. Define a linear map $\phi : U \to (U+W)/W$ by $\phi(u) = u + W$.
   (*a*) Use the first isomorphism theorem, applied to ϕ, to prove the second isomorphism theorem:

$$U/(U \cap W) \cong (U+W)/W.$$

   (*b*) Deduce that, when V is finite-dimensional,

$$\dim(U+W) = \dim U + \dim W - \dim(U \cap W).$$

Please hand in at 4W level 1 by NOON on Thursday 31st October 2024