MA22020: Exercise sheet 1

Warmup questions

1. Let U be a subset of a vector space V. Show that U is a linear subspace of V if and only if U satisfies the following conditions:
   (i) $0 \in U$;
   (ii) For all $u_1, u_2 \in U$ and $\lambda \in \mathbb{F}$, $u_1 + \lambda u_2 \in U$.

2. Which of the following subsets of $\mathbb{R}^3$ are linear subspaces? In each case, briefly justify your answer.
   (a) $U_1 := \{(x_1, x_2, x_3) \mid x_1^2 + x_2^2 + x_3^2 = 1\}$ (b) $U_2 := \{(x_1, x_2, x_3) \mid x_1 = x_2\}$ (c) $U_3 := \{(x_1, x_2, x_3) \mid x_1 + 2x_2 + 3x_3 = 0\}$

3. Which of the following maps $f : \mathbb{R}^2 \to \mathbb{R}^2$ are linear? In each case, briefly justify your answer.
   (a) $f(x, y) = (5x + y, 3x - 2y)$ (b) $f(x, y) = (5x + 2, 7y)$ (c) $f(x, y) = (\cos y, \sin x)$ (d) $f(x, y) = (3y^2, x^3)$.

4. Let $U, W \leq V$ be subspaces of a vector space V.
   When is $U \cup W$ also a subspace of V?

Homework

5. Which of the following subsets of $\mathbb{C}^3$ are linear subspaces over $\mathbb{C}$? In each case, briefly justify your answer.
   (a) $U_1 := \{(z_1, z_2, z_3) \mid z_1 z_2 = 1\}$ (b) $U_2 := \{(z_1, z_2, z_3) \mid z_1 = \bar{z}_2\}$ (c) $U_3 := \{(z_1, z_2, z_3) \mid z_1 + \sqrt{-1} z_2 + 3z_3 = 0\}$

6. Let V, W be vector spaces, $v_1, \ldots, v_n$ a basis of V and $w_1, \ldots, w_n$ a list of vectors in W. Let $\phi : V \to W$ be the unique linear map with

$$\phi(v_i) = w_i,$$

   for all $1 \leq i \leq n$. Show:
   (a) ϕ injects if and only if $w_1, \ldots, w_n$ is linearly independent.
   (b) ϕ surjects if and only if $w_1, \ldots, w_n$ spans W.
   Deduce that ϕ is an isomorphism if and only if $w_1, \ldots, w_n$ is a basis for W.

Please hand in at 4W level 1 by NOON on Thursday 16th October 2025

MA22020: Exercise sheet 1—Solutions

1. First suppose that $U \leq V$. The U is non-empty so there is some $u \in U$ and then, since U is closed under addition and scalar multiplication, $0 = u + (-1)u \in U$ also and condition (i) is satisfied. Now if $u_1, u_2 \in U$ and $\lambda \in \mathbb{F}$, then $\lambda u_2 \in U$ (U is closed under scalar multiplication) and so $u_1 + \lambda u_2 \in U$ (U is closed under addition). Thus condition (ii) holds also.
   For the converse, if conditions (i) and (ii) hold, then, first, $0 \in U$ so U is non-empty and, second, U is closed under addition (take $\lambda = 1$ in condition (ii)) and under scalar multiplication (take $u_1 = 0$ in condition (ii)). Thus $U \leq V$.

2. (*a*) U_1 is not a subspace as it does not contain 0!
   (*b*) U_2 is a subspace: in fact, it is $\ker \phi_A$ where $A = \begin{pmatrix} 1 & -1 & 0 \end{pmatrix}$.
   (*c*) U_3 is a subspace. It is $\ker \phi_A$ for $A = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$.

3. (*a*) Here f is linear: it is the map ϕ_A corresponding to the matrix
$$A = \begin{pmatrix} 5 & 1 \\ 3 & -2 \end{pmatrix}.$$
   (*b*) This is not linear (because of that $+2$ term). In particular $f(0,0) = (2,0) \neq 0$!
   (*c*) Again $f(0,0) = (1,0) \neq 0$ so this f cannot be linear. Of course, we already *know* this because it is certainly not true that $\cos(y_1 + y_2) = \cos y_1 + \cos y_2$.
   (*d*) Another non-linear map: for example $f(2x, 2y) \neq 2f(x,y)$.

4. If $U \subseteq W$ then $U \cup W = W$ is a subspace and similarly if $W \subseteq U$. In any other case, $U \cup W$ is not a subspace: we can find $u \in U \setminus W$ and $w \in W \setminus U$ and then $u + w \notin U$ (else $w = (u + w) - u \in U$) and similarly $u + w \notin W$. Thus $U \cup W$ is not closed under addition.

5. (*a*) $0 \notin U_1$ so U_1 is not a subspace.
   (*b*) U_2 is not a subspace because it is not closed under complex scalar multiplication: $(1,1,0) \in U_2$ but $i(1,1,0) = (i,i,0)$ is not (here $i = \sqrt{-1}$). In general, any time you see complex conjugation in the definition of a subset, it is unlikely to be a complex subspace.
   (*c*) $U_3 = \ker \phi_A$ for $A = \begin{pmatrix} 1 & \sqrt{-1} & 3 \end{pmatrix}$ and so is a subspace.

6. (*a*) $\lambda_1 w_1 + \cdots + \lambda_n w_n = 0$ if and only if $\lambda_1 v_1 + \cdots + \lambda_n v_n \in \ker \phi$. Thus $w_1, \ldots, w_n$ is linearly independent if and only if ϕ has trivial kernel.
   (*b*) ϕ surjects if and only if any $w \in W$ can be written $w = \phi(v)$, or equivalently,
$$w = \phi(\lambda_1 v_1 + \cdots + \lambda_n v_n) = \lambda_1 w_1 + \cdots + \lambda_n w_n,$$
   for some λ_i, $1 \leq i \leq n$.