Section A

1. Let $U_1, U_2, U_3 \leq \mathbb{R}^3$ be given by:

$$U_1 = \text{span}\{(1,0,-1)\};$$
$$U_2 = \text{span}\{(1,1,1),(1,-1,0)\};$$
$$U_3 = \{x \in \mathbb{R}^3 \mid x_2 + x_3 = 0\}.$$

Which of the three sums $U_i + U_j$, $1 \leq i < j \leq 3$, are direct? In each case, briefly justify your answer. [4]

2. Let $U = \text{span}\{(0,1,2)\} \leq \mathbb{R}^3$. Define subspaces $W_1, W_2 \leq \mathbb{R}^3/U$ by

$$W_1 = \text{span}\{(1,1,1) + U\}, \qquad W_2 = \text{span}\{(2,4,6) + U\}.$$

Is $W_1 + W_2$ direct? Justify your answer. [4]

3. Let $p = (x-7)^2(x-3) \in \mathbb{R}[x]$ and let A be a square matrix such that $p(A) = 0$.
What are the possibilities for the minimum polynomial of A?
How does your answer change if you know that A is not diagonal? [4]

4. Let A be given by

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}.$$

Find a Jordan basis for A. [4]

5. Let $q\colon \mathbb{R}^2 \to \mathbb{R}$ be the quadratic form given by $q(x) = 4x_1^2 - 4x_1x_2 + x_2^2$. Find the rank and signature of q. [4]

Section B

6. (a) Let U be a linear subspace of a possibly infinite-dimensional vector space V. Suppose that the quotient space V/U is finite-dimensional. Show that U has a complement in V. [6]

   (b) Let A be given by

$$\begin{pmatrix} 2 & -1 & 0 \\ 6 & -3 & -2 \\ -6 & 3 & 4 \end{pmatrix}.$$

   (i) Find the characteristic and minimum polynomials of A.

   (ii) Find the Jordan normal form of A.

[9]

7. (a) Let $B \in M_3(\mathbb{R})$ be given by

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & -2 & 1 \\ 1 & -3 & 1 \end{pmatrix}$$

   (i) Show that the minimum polynomial of B is x^3.

   (ii) Can the following basis of $\mathbb{R}^3$ be re-ordered to give a Jordan basis for B? If so, how? If not, find a Jordan basis for B.

$$(0,0,1), \qquad (0,1,1), \qquad (2,2,4).$$

[10]

   (b) Contemplate the symmetric bilinear form B_A on $\mathbb{R}^5$ where

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & -1 & 6 & 7 & 8 \\ 3 & 6 & 0 & 9 & 10 \\ 4 & 7 & 9 & -3 & 11 \\ 5 & 8 & 10 & 11 & 4 \end{pmatrix}.$$

   Let (p,q) be the signature of B_A. Show that $p,q > 0$. [5]