DEPARTMENT OF MATHEMATICAL SCIENCES
Outline solution to Examination Questions

UNIVERSITY OF BATH

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1. U_1+U_2 direct unless $U_1 \leq U_2$ since $U_1 \cap U_2 = \{0\}$ or U_1. $U_1 \leq U_2$ iff
$(1,0,-1) = \lambda(1,1,1) + \mu(1,-1,0)$ some λ, μ
i.e. $\lambda = \mu = 0$ $\lambda+\mu = 1$ $\lambda = -1$: incompatible
$\therefore$ U_1+U_2 direct.
U_1+U_3 direct unless $U_1 \leq U_3$ but
$0-1 \neq 0$ $\therefore$ $U_1 \not\leq U_3$ so U_1+U_3 direct
U_2+U_3 not direct else $\dim U_2+U_3 = 2+2$
$= 4 > 3$

Alternatively $\lambda(1,1,1)+\mu(1,-1,0) \in U_3$
iff $(\lambda-\mu)+\lambda = 0$ so
$\lambda = 1$ $\mu = 2$ gives
$(3,-1,1) \in U_2 \cap U_3$
so sum not direct.

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2. Either $W_1 \cap W_2 = \{0\}$ or $W_1 = W_2$

The latter holds iff $\exists \lambda \in \mathbb{R}$ s.t

$$\lambda(1,1,1) + U = (2,4,6) + U \quad \text{i.e.}$$

$$(\lambda-2, \lambda-4, \lambda-6) = \mu(0,1,2) \text{ some } \mu \in \mathbb{R}$$

This gives $\lambda = 2$ then

$$(0,-2,-4) = \mu(0,1,2) \text{ so}$$

$\mu = -2$ $\therefore$ $W_1 = W_2$ and sum not direct.

[4]

3. $m_A \mid p$ so m_A could be

$(x-3)$, $(x-7)$, $(x-7)^2$, $(x-7)(x-3)$ [3]

or $(x-7)^2(x-3)$.

If A not diag it is not multiple of I

so drop $(x-3)$ and $(x-7)$. [1]

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4. Only eigenvalue is 1 ✓

$\operatorname{Ker} A - I = \operatorname{span}\{(1,0)\}$

So see $\underline{v}$ s.t. $(A-I)v = (1,0)$

$$\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

So $\begin{pmatrix} 0 \\ 1/2 \end{pmatrix}$ will do ✓

$(1,0), (0,1/2)$ is Jordan basis.

[4]

5. $4x_1^2 - 4x_1x_2 + x_2^2 = (2x_1 - x_2)^2$

So rank $= 1$ sig $= (1,0)$

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(a) Let $q: V \to V/U$ be quotient map & $w_1, \dots, w_n$ be basis of V/U.

Choose $v_1, \dots, v_n \in V$ with $q(v_i) = w_i$ & set $W = \text{span}\{v_1, \dots, v_n\}$. [2]

Claim $V = U \oplus W$.

- $V = U + W$: let $v \in V$ then
$$q(v) = \sum \lambda_i w_i \text{ some } \lambda_i$$
$$= q\left(\sum \lambda_i v_i\right) \text{ so } v - \sum \lambda_i v_i \in \ker q = U$$

$$\therefore \quad v = \underbrace{\sum \lambda_i v_i}_{\in W} + \underbrace{\left(v - \sum \lambda_i w_i\right)}_{\in U}$$ [2]

- $U \cap W = \{0\}$

Let $\sum \lambda_i v_i \in U \cap W$ then
$$0 = q\left(\sum \lambda_i v_i\right) = \sum \lambda_i w_i \Rightarrow \lambda_i = 0 \ \forall i \Rightarrow \sum \lambda_i v_i = 0$$
since $w_1, \dots, w_n$ basis [2]

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(b) (i)

$$\begin{vmatrix} 2-x & -1 & 0 \\ 6 & -3-x & -2 \\ -6 & 3 & 4-x \end{vmatrix} = (2-x)\begin{vmatrix} -(3+x) & -2 \\ 3 & 4-x \end{vmatrix}$$

$$+ \begin{vmatrix} 6 & -2 \\ -6 & 4-x \end{vmatrix} =$$

$$(2-x)\{(x-4)(x+3)+6\} + 24-6x-12 =$$

$$(2-x)\,(x^2-x-6) + 6(2-x) =$$

$$(2-x)\,x(x-1) = \Delta_A$$

$$\therefore \quad m_A = -\Delta_A = x(x-1)(x-2)$$ [6]

(ii) The eigenvals are $\therefore$ 0, 1, 2

& largest Jordan block is 1x1

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$\therefore$ JNF is $J(0,1) \oplus J(1,1) \oplus J(2,1)$

$\therefore$ A is diagonalisable. [3]

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(a) (i)

$$\Delta_B = \begin{vmatrix} 1-x & -1 & 0 \\ 0 & -2-x & 1 \\ 1 & -3 & 1-x \end{vmatrix} =$$

$$(1-x)\left((2+x)(x-1)+3\right) - 1 =$$

$$(1-x)(x^2+x+1) - 1 =$$

$$x^2+x+1-x^3-x^2-x-1 = -x^3$$

$\therefore$ $M_B = x^2$ or x^3 (not x since not zero)

But $B^2 =$

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & -2 & 1 \\ 1 & -3 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 & 0 \\ 0 & -2 & 1 \\ 1 & -3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & * \\ & \end{pmatrix} \neq 0$$

$\therefore$ $M_B = x^3$

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(ii) Jordan basis is v_1, v_2, v_3 with each $B\underline{v}_i = \underline{v}_{i-1}$ or 0

$$B\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\1\\1\end{pmatrix}$$

$$B\begin{pmatrix}0\\1\\1\end{pmatrix} = \begin{pmatrix}-1\\-1\\-2\end{pmatrix} \neq \begin{pmatrix}2\\2\\4\end{pmatrix}$$

So given basis cannot be reordered to Jordan basis but

$$B\begin{pmatrix}-1\\-1\\-2\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} \text{ so}$$

$(-1,-1,-2), (0,1,1), (0,0,1)$ is a Jordan basis.

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(b) Let $U_1 = \mathrm{span}\{e_1\}$ then

for $v = \lambda e_1 \in U_1$

$$B_A(v,v) = \lambda^2 B_A(e_1,e_1) = \lambda^2$$

so B_A pos. def on U_1

$\therefore$ $p = \max\{\dim U \mid B_A|_{U\times U} \text{ pos def}\}$
$\geq \dim U_1 = 1$

Similarly, let $U_2 = \mathrm{span}\{e_2\}$

then $w = \lambda e_2$ has $B_A(w,w) = -\lambda^2$

so B_A neg. def on U_2

$\therefore$ $q \geq 1$ also

[5]

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