DEPARTMENT OF MATHEMATICAL SCIENCES
Outline solution to Examination Questions

UNIVERSITY OF BATH

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1. Compute $U \cap W$:

$U = \text{span}\{(1,1,1),(1,2,1)\} = \{(1,1,1),(0,1,0)\}$

$= \{(\lambda, \lambda+\mu, \lambda) \mid \lambda, \mu \in \mathbb{R}\}$

$W = \{(x,0,y) \mid x,y \in \mathbb{R}\}$

$\therefore (\lambda, \lambda+\mu, \lambda) \in U \cap W \iff \lambda+\mu = 0$

so of form $(\lambda, 0, \lambda)$.

$\therefore$ basis is $(1,0,1)$ [2]

$U+W = \mathbb{R}^3$ else $U+W = U$ ※

so any basis of $\mathbb{R}^3$ will do. [2]

2. Sum $\ker\phi \oplus \operatorname{im}\phi$ direct so

$\dim \ker\phi \oplus \operatorname{Im}\phi = \operatorname{rank}\phi + \operatorname{nullity}\phi$

$= \dim V$ by rank-nullity

$\therefore \ker\phi \oplus \operatorname{Im}\phi = V$. [4]

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3. $A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 2 \\ 1 & 0 & 0 \end{pmatrix}$ so $A^2 = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 2 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 2 \\ 1 & 0 & 0 \end{pmatrix}$

$$= \begin{pmatrix} 1 & 0 & 2 \\ 2 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

and $(3,1)$ slot of $a_0 I_3 + a_1 A + A^2 = 0$ gives

$0 = 1$ $\#$ so $\deg m_A > 2$.

$$A^3 = \begin{pmatrix} 1 & 0 & 2 \\ 2 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 2 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 2 & 2 \\ 1 & 0 & 2 \end{pmatrix}$$

$$= 2I_3 + A$$

$\therefore$ $m_A = x^3 - x - 2$ [4]

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4. $\deg \Delta_\phi = 6$ so $\dim V = 6$

max size of 3-Jordan block is 2

max size of 2-Jordan block is 1

so $G_\phi(2) = E_\phi(2)$ 2-dim'l

~~so poss~~ while $\dim G_\phi(3) = 4$

$\therefore$ two possibilities:

$$J(3,2) \oplus J(3,2) \oplus J(2,1) \oplus J(2,1)$$

or

$$J(3,2) \oplus J(3,1) \oplus J(3,1) \oplus J(2,1) \oplus J(2,1)$$

[4]

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5. $(x_1, x_2, x_3) \in \text{sol}\,\cancel{}\,E$ iff

$$x_1 + 2x_2 - x_3 = 0$$

$\therefore$ $\text{sol}\,E = \{(x_1, x_2, x_1+2x_2) \mid x_1, x_2 \in \mathbb{R}\}$

$= \text{span}\{\underbrace{(1,0,1), (0,1,2)}_{\text{basis}}\}$ [4]

6. rank: $\dim V - \dim\{v \in V \mid B(v,w) = 0 \; \forall w \in V\}$

sig: (p,q) $\quad p = \max\{\dim U \mid U \leq V, \; B|_{U\times U} \text{ pos. def}\}$

$q = \max\{\dim W \mid W \leq V, \; B|_{W\times W} \text{ neg-def}\}$

Sylvester: rank $= p+q$

$p = \#\{\text{pos. entries in diag-matrix representing } B\}$

$q = \#\{\text{neg. entries " " " " }\}$ [4]

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(a) Let $v_1+v_2 \in \ker\phi_{|V_1} \oplus \ker\phi_{|V_2}$ so

$\phi(v_1) = \phi(v_2) = 0$ $\therefore$ $\phi(v_1+v_2) = 0$ i.e.

$v_1+v_2 \in \ker\phi$. Conversely, if $v \in \ker\phi$

write $v = v_1+v_2$ with $v_i \in V_i$

then $0 = \phi(v) = \phi(v_1)+\phi(v_2)$ with $\phi(v_i) \in V_i$

$= 0 + 0$

$\therefore$ since $V_1 \oplus V_2$ is direct sum $\phi(v_i) = 0$ $i=1,2$

or $v \in \ker\phi_{|V_1} \oplus \ker\phi_{|V_2}$. [6]

(b) (i) Define $\bar\phi : V/U \to V/U$ by

$\bar\phi(q(v)) = q(\phi(v))$

- well-defined: if $q(v) = q(v')$ then

$v - v' \in \ker q = U$ $\therefore$ $\phi(v-v') \in U$

so $q(\phi(v-v')) = 0$ $\therefore$ $q(\phi(v)) = q(\phi(v'))$

by linearity of $q \circ \phi$.

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- $\bar{\phi}$ linear:

$$\bar{\phi}(q(v)+\lambda q(w)) \underset{\uparrow\ q \text{ linear}}{=} \bar{\phi}(q(v+\lambda w))$$

$$\underset{\text{def of } \bar{\phi}}{=} q(\phi(v+\lambda w)) = q(\phi(v)) + \lambda q(\phi(w))$$

$$= \bar{\phi}(q(v)) + \lambda \bar{\phi}(q(w)) \checkmark.$$

[6]

(c) It suffices to show that $u_1, \dots, u_k, v_1, \dots, v_{n-k}$ lin. indep since $\dim V = n$

If $\lambda_1 u_1 + \dots + \lambda_k u_k + \mu_1 v_1 + \dots + \mu_{n-k} v_{n-k} = 0$

take q of this to get

$$0 + \mu_1 q(v_1) + \dots + \mu_{n-k} q(v_{n-k}) = 0$$

$\Rightarrow \mu_i = 0 \quad 1 \le i \le n-k$ since $q(v_i)$ lin. indep.

$\therefore \lambda_1 u_1 + \dots + \lambda_k u_k = 0 \Rightarrow \lambda_j = 0 \quad 1 \le j \le k$

since the u_i lin. indep.

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(a) Since $m_A \mid \Delta_A$ either $m_A = (x-2)(x+1)$ or $m_A = (x-2)(x+1)^2$.

Check $(x-2)(x+1)$:

$$(A-2I) = \begin{pmatrix} -1 & -1 & 2 \\ -3 & -3 & -3 \\ 1 & 1 & -2 \end{pmatrix} \qquad (A+I) = \begin{pmatrix} 2 & -1 & 2 \\ -3 & 0 & -3 \\ 1 & 1 & 1 \end{pmatrix}$$

$$\therefore (A-2I)(A+I) = \begin{pmatrix} 3 & & \\ & * & \\ & & \end{pmatrix} \neq 0$$

$$\therefore m_A = (x-2)(x+1)^2$$ [5]

(b) From m_A we see that $\exists$ at least one $J(-1,2)$ & $J(2,1)$

$\therefore$ JNF $= J(-1,2) \oplus J(2,1)$ [4]

(c) For Jordan basis, first compute $G_A(-1) = \ker(A+I)^2$

$$(A+I)^2 = \begin{pmatrix} 2 & -1 & 2 \\ -3 & 0 & -3 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & -1 & 2 \\ -3 & 0 & -3 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 9 & 0 & 9 \\ -9 & 0 & -9 \\ 0 & 0 & 0 \end{pmatrix}$$

$\ker(A+I)^2 = \{(x,y,-x) \mid x,y \in \mathbb{R}\}$

$\ker(A+I) = \{(x,y,-x) \mid 2x-y-2x=0 \text{ ie } y=0\} = \operatorname{span}\{(1,0,-1)\}$

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Now need $x \in \ker(A+I)^2$ s.t. $(A+I)\underline{x} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ & $\underline{x} = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$ clearly works

$\therefore$ Jordan basis of $G_A(-1) = (1,0,-1), (0,-1,0)$

Finally $x \in \ker(A-2I)$ if $\begin{array}{l} -x_1 - x_2 + 2x_3 = 0 \\ x_1 + x_2 + x_3 = 0 \end{array} \Rightarrow 3x_3 = 0 \Rightarrow x_3 = 0$

& then $x_1 = -x_2$ $\therefore$ $(1,-1,0)$ is basis.

Thus Jordan basis is

$(1,0,-1), (0,-1,0), (1,-1,0)$.

[9]

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(a) (i) No: rank $= 4+2 = 6 > 5$ ※ [2]

(ii) No: $p, q \geq 0$ being dimensions [2]

(iii) Yes: eg $q(x) = x_1^2 + x_2^2 - x_3^2$ [2]

(iv) Yes: eg $q(x) = x_1^2 + x_2^2 + x_3^2 - x_4^2 - x_5^2$ [2]

(b) Find a diag basis for A_t & use it as cols of P_t:

Exploit zero in $(2,1)$ slot to see that e_1, e_2 are start of diag. basis. So need $y = v_3$ s.t.

$$(1\ 0\ 0) A_t \underline{y} = (1\ 0\ 0)\begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & t \\ -1 & t & 0 \end{pmatrix} \underline{y} = (1\ 0\ -1)\underline{y} = y_1 - y_3 = 0$$

$$(0\ 1\ 0)\begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & t \\ -1 & t & 0 \end{pmatrix} \underline{y} = (0\ 2\ t)\underline{y} = 2y_2 + t y_3 = 0$$

$\therefore$ $v_3 = (2\ -t\ 2)$ will do

$\therefore$ $P_t = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -t \\ 0 & 0 & 2 \end{pmatrix}$ has $P_t^T A_t P_t$ diagonal. [10]

Total