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1. Show $U \cap W = \{0\}$

Suppose $(1,1,1) = \lambda(1,0,-1) + \mu(2,-1,0)$ so

$$1 = \lambda + 2\mu = -\lambda = -\mu$$

then $1 = -3$ # $\therefore (1,1,1) \notin U$ so $U \cap W = \{0\}$

$\therefore$ sum direct.

[4]

2. $u = \lambda(1,0,-1) + \mu(2,-1,0)$

$w = \nu(1,1,1)$

$\therefore$ need $(1,2,3) = (\lambda + 2\mu + \nu, -\mu + \nu, \nu - \lambda)$

$\therefore \quad \mu = \nu - 2 \quad \lambda = \nu - 3$

$1 = \nu - 3 + 2\nu - 4 + \nu = 4\nu - 7$ i.e. $\nu = 2$

$\therefore \quad u = (-1,0,1) \quad w = (2,2,2)$ [4]

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3. $W^{\perp} = \{ (x,y,z) \in \mathbb{R}^3 \mid \underbrace{(x,y,z) \cdot (1,1,1)}_{= x+y+z} = 0 \}$

$\dim W^{\perp} = 3 - \dim W = 2$

so basis is, for example, $(1,-1,0), (1,0,-1)$
[there are many others]

[4]

4. Let $\phi(v) = \lambda v$ with $v \neq 0$

Then $\langle \phi(v), \phi(v) \rangle = \langle v, v \rangle$

$\langle \phi(v), \phi(v) \rangle = \langle \lambda v, \lambda v \rangle = |\lambda|^2 \langle v, v \rangle$ & $\langle v, v \rangle \neq 0$

$\therefore |\lambda|^2 = 1$

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5. $v - w \neq 0$ $\therefore$ by sufficiency principle $\exists \alpha \in V^*$ s.t. $\alpha(v-w) \neq 0$ i.e.

$$\alpha(v) - \alpha(w) \neq 0.$$

[4]

6. Complete the square: $x_1^2 - 2x_1x_2 + t x_2^2 =$

$$(x_1 - x_2)^2 + (t-1)x_2^2$$

Since $x_1 - x_2$ & x_2 are lin. indep, Q_t has rank 1 exactly when $t = 1$.

[4]

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7. (a) (i) Let $v \in V$ then

$$v = \mathrm{id}_V(v) = \pi_1(v) + \pi_2(v) + \pi_3(v)$$

$\therefore$ $V = \mathrm{im}\,\pi_1 + \mathrm{im}\,\pi_2 + \mathrm{im}\,\pi_3$

To see that sum is direct, let

$$v = w_1 + w_2 + w_3 \quad \text{with} \quad w_i \in \mathrm{im}\,\pi_i$$

we show that $w_i = \pi_i(v)$.

Now $w_j = \pi_j(u_j)$ some $u_j \in V$ so

$$\pi_i(w_j) = \pi_i\pi_j(u_j) = 0 \quad \text{if } i \neq j$$

$\therefore$ $\pi_i(v) = \pi_i(w_i) = \pi_i(\pi_i(u_i)) = \pi_i(u_i) = w_i$
as required. [6]

(ii) We have seen that $\pi_1\pi_j(u) = 0$ $j \neq 1$

so $\mathrm{Im}\,\pi_2, \mathrm{Im}\,\pi_3 \leq \mathrm{Ker}\,\pi_1$

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Conversely if $v \in \ker \tilde{\pi}_1$

$v = \cancel{\tilde{\pi}_1(v)} + \tilde{\pi}_2(v) + \tilde{\pi}_3(v) \in \operatorname{Im}\tilde{\pi}_2 + \operatorname{Im}\tilde{\pi}_2$

$\therefore \quad \ker \tilde{\pi}_1 = \operatorname{Im}\tilde{\pi}_2 + \operatorname{Im}\tilde{\pi}_3$

Finally if $v = \tilde{\pi}_2(u) = \tilde{\pi}_3(w) \in \operatorname{Im}\tilde{\pi}_2 \cap \operatorname{Im}\tilde{\pi}_3$

then $\tilde{\pi}_2(v) = \tilde{\pi}_2\tilde{\pi}_2(u) = \tilde{\pi}_2\tilde{\pi}_3(w) = 0$

$\tilde{\pi}_2\tilde{\pi}_2(u) = \tilde{\pi}_2(u) = v$

so $\operatorname{im}\tilde{\pi}_2 \cap \operatorname{im}\tilde{\pi}_3 = \{0\}$ & sum is direct.

$\therefore \quad \ker \tilde{\pi}_1 = \operatorname{im}\tilde{\pi}_2 \oplus \operatorname{im}\tilde{\pi}_3.$ [6]

(b) Well-defined: if $q(v) = q(w)$

$v - w \in \ker q = U \quad \therefore \quad \phi(v-w) \in U$

$\therefore \quad q \circ \phi(v-w) = 0$ i.e. $q(\phi(v)) = q(\phi(w))$

$\therefore \quad \tilde{\phi}$ well-defined

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Linear: $\tilde{\varphi}(q(v)+\lambda q(w)) \underset{q \text{ linear}}{=}$

$$\tilde{\varphi}(q(v+\lambda w)) \underset{\text{def}}{=} q\varphi(v+\lambda w) \underset{q\varphi \text{ linear}}{=} q\varphi(v)+\lambda q\varphi(w)$$

$$= \tilde{\varphi}(q(v))+\lambda\tilde{\varphi}(q(w)).$$

[6]

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8. (a) There are several arguments. Here is one: for $\psi \in L(V)$

$$\operatorname{rank}\psi = \operatorname{rank}\psi^*$$

[Apply rank-nullity to $\ker\psi = (\operatorname{im}\psi^*)^\perp$:

$v \in \ker\psi \iff \langle \psi(v), w\rangle = 0 \ \forall w \iff \langle v, \psi^*(w)\rangle = 0 \ \forall w \iff v \in (\operatorname{im}\psi^*)^\perp$]

Apply this to $\psi = \phi - \lambda \mathrm{id}_V$ then $\psi^* = \phi^* - \bar{\lambda}\mathrm{id}_V$ & λ is eval of ϕ iff $\operatorname{rank}\psi < \dim V$ iff $\operatorname{rank}\psi^* < \dim V$ iff $\bar{\lambda}$ is eval of ϕ^*. [6]

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(b) (i) $U^{\perp} = \left\{ (x_1, x_2, x_3, x_4) \,\middle|\, \begin{matrix} x_1 + x_2 - x_3 - x_4 = 0 \\ x_1 - x_4 = 0 \end{matrix} \right\}$

$= \left\{ (x_1, x_2, x_3, x_4) \,\middle|\, \begin{matrix} x_1 = x_4 \\ x_2 = x_3 \end{matrix} \right\} =$

$= \{ (\lambda, \mu, \mu, \lambda) \mid \lambda, \mu \in \mathbb{R} \}$ [4]

$= \operatorname{span} \{ (1,0,0,1), (0,1,1,0) \}$

(ii) $\dim U^{\perp} = 2$ (spanned by two lin. indep vectors)

$\therefore$ basis is $(1,0,0,1), (0,1,1,0)$

G-S: $w_1 = (1,0,0,1)$ $\quad \|w_1\|^2 = 2$

$\therefore u_1 = \frac{1}{\sqrt{2}}(1,0,0,1)$

$\langle w_1, v_2 \rangle = 0$ so

$w_2 = v_2$ $\quad$ & $\quad u_2 = \frac{1}{\sqrt{2}}(0,1,1,0)$

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(iii) Orthoproj onto $U^\perp$:

$$\pi_{U^\perp}(v) = \langle u_1, v\rangle u_1 + \langle u_2, v\rangle u_2$$
$$= \frac{\langle w_1, v\rangle w_1}{\|w_1\|^2} + \frac{\langle w_2, v\rangle w_2}{\|w_2\|^2}$$

$\circ$ With $v = (1, 2, 3, 1)$

$\therefore$ $\langle w_1, v\rangle = 2 \quad \langle w_2, v\rangle = 5$

$$\pi_{U^\perp}(v) = \frac{2}{2}(1,0,0,1) + 5/2\,(0,1,1,0)$$
$$= (1, 5/2, 5/2, 1)$$

$$\pi_U(v) = v - \pi_{U^\perp}(v)$$
$$= (1,2,3,1) - (1, 5/2, 5/2, 1)$$
$$= (0, -1/2, 1/2, 0)$$

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9. (a) $Q = \alpha\beta = \frac{1}{4}((\alpha+\beta)^2 - (\alpha-\beta)^2)$

and $\alpha+\beta$, $\alpha-\beta$ are lin. indep since α, β are. Thus Q has sig $(1,1)$ or rank 2.

[6]

(b) We find a diagonalising basis.
Exploit the zero in $(3,1)$ slot to see that $v_1 = e_1$ $v_2 = e_3$ is start of such a basis.
$\therefore$ need $v_3 = y$ with

$$B_{A_t}(e_1, y) = B_{A_t}(e_2, y) = 0$$

i.e. $$(1,0,0)\begin{pmatrix} 1 & 2 & 0 \\ 2 & 0 & t \\ 0 & t & 3 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = y_1 + 2y_2 = 0$$

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$$(0,0,1)\begin{pmatrix}1&2&0\\2&0&t\\0&t&3\end{pmatrix}\begin{pmatrix}y_1\\y_2\\y_3\end{pmatrix} = ty_2+3y_3 = 0$$

$\therefore$ $y=(-6,3,-t)$ will do γ

$$B_{A_t}(y,y) = (-6,3,-t)\begin{pmatrix}1&2&0\\2&0&t\\0&t&3\end{pmatrix}\begin{pmatrix}-6\\3\\-t\end{pmatrix}$$

$$= (-6,3,-t)\begin{pmatrix}0\\-12-t^2\\0\end{pmatrix} = -3(12+t^2)$$

$\therefore$ A_t congruent to

$$\begin{pmatrix}1&&0\\&3&\\0&&-3(12+t^2)\end{pmatrix}$$

which has sig $(2,1)$ for all t.

$\therefore$ no $t\in\mathbb{R}$ such that sig $=(3,0)$ [12]

Total