Section A

1. Define subspaces $U, W \leq \mathbb{R}^3$ by:

$$U = \text{span}\{(1,1,1),(1,2,0)\}, \qquad W = \text{span}\{(1,0,2)\}\,.$$

Is the sum $U + W$ direct? (You must justify your answer.) [4]

2. With $U \leq \mathbb{R}^3$ as in question 1, $v = (1,2,3)$ and $w = (0,2,1)$, is $v \equiv w \mod U$? (You must justify your answer.) [4]

3. For $x, y \in \mathbb{C}^3$, let

$$\langle x, y \rangle = x_1 y_1 + x_2 y_2 + \bar{x}_3 y_3.$$

Is $\langle\ ,\ \rangle$ an inner product on $\mathbb{C}^3$? (You must justify your answer.) [4]

4. Show that $A \in M_{2\times 2}(\mathbb{C})$ given by

$$\frac{1}{\sqrt{5}}\begin{pmatrix} 1 & 2i \\ 2i & 1 \end{pmatrix}$$

is a unitary matrix and compute its eigenvalues. [4]

5. With $U \leq \mathbb{R}^3$ as in question 1, write down a non-zero element of $\operatorname{ann} U$.
Hint: any $\alpha \in (\mathbb{R}^3)^*$ is of the form $\alpha(x) = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3$. [4]

6. Define a quadratic form Q on $\mathbb{R}^2$ by

$$Q(x) = 4x_1^2 + 4x_1 x_2 + x_2^2.$$

Compute the rank and signature of Q. [4]

Section B

7. Let V be a vector space over a field $\mathbb{F}$.

(a) Let $V_1, \dots, V_k \leq V$ be subspaces of V.
Show that the sum $V_1 + \cdots + V_k$ is direct if and only if, whenever $v_i \in V_i$, $1 \leq i \leq k$, satisfy

$$v_1 + \cdots + v_k = 0$$

then each $v_i = 0$. [6]

(b) Let $u, w \in V$ and $V_1, V_2 \leq V$ subspaces such that $u + V_1 = w + V_2$.
Prove that $V_1 = V_2$. [6]

(c) Let $\alpha \in V^*$ be non-zero. Prove that

$$\dim(V/\ker\alpha) = 1.$$

[6]

8. (a) Let a, b, c, d, e be positive real numbers. Show that

$$25 \leq (a+b+c+d+e)(1/a+1/b+1/c+1/d+1/e).$$

For what a, b, c, d, e do we get equality? [5]

(b) Find the QR decomposition for the matrix A given by

$$A = \begin{pmatrix} 1 & 1 & 3 \\ -1 & 0 & 1 \\ 0 & -1 & -1 \end{pmatrix}.$$

[7]

(c) Let $U = \text{span}\{(1,\text{-}1,0),(1,0,\text{-}1)\} \leq \mathbb{R}^3$ and $v = (2,1,2)$.
Find the closest point of U to v (where we define distance in $\mathbb{R}^3$ using the dot product). [6]

9. Let B be the symmetric bilinear form on $\mathbb{R}^3$ defined by the matrix

$$A = \begin{pmatrix} -3 & 2 & -2 \\ 2 & -1 & 1 \\ -2 & 1 & -1 \end{pmatrix}.$$

Thus $B(x, y) = \mathbf{x}^T A \mathbf{y}$.

(a) Find a diagonalising basis for B. [10]

(b) Compute the rank and signature of B. [4]

(c) Compute the radical of B. [4]