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Outline solution to Examination Questions

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1. $U+W$ direct if $U \cap W = \{0\}$

If non-zero $\exists\, \lambda, \mu$ st

$$(1,0,2) = \lambda(1,1,1) + \mu(1,2,0)$$

$\Rightarrow \lambda = 2$ $\quad \lambda+\mu = 1$, $\lambda + 2\mu = 0$ so $\mu = -1$ is sol^{n}

i.e. $2(1,1,1) - (1,2,0) = (1,0,2) \in U \cap W$

$\therefore$ sum not direct. [4]

2. $v \equiv w \mod U$ iff

$v - w = (1,0,2) \in U$ which

holds by Q1 [4]

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3. Not an inner product for many reasons.
Eg: not definite: $x = (i, 0, 0)$
$\langle x, x \rangle = -1$
(Also not antilinear in 1^{st} slot; not conj. symmetric ...)

4. $$A^\dagger A = \frac{1}{5}\begin{pmatrix} 1 & -2i \\ -2i & 1 \end{pmatrix}\begin{pmatrix} 1 & 2i \\ 2i & 1 \end{pmatrix}$$
$$= \frac{1}{5}\begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \therefore$$
A unitary. [1]

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Char poly of $\sqrt{5}A$

$$\begin{vmatrix} 1-\lambda & 2i \\ 2i & 1-\lambda \end{vmatrix} = (1-\lambda)^2 + 4$$

$$= 5 - 2\lambda + \lambda^2$$

$$\therefore \quad \lambda = \frac{2 \pm \sqrt{4-20}}{2} = \frac{2 \pm 4i}{2}$$

$$= 1 \pm 2i$$

$\therefore$ evals $1 \pm 2i / \sqrt{5}$ [3]

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5. We need $\alpha(1,1,1) = \alpha_1 + \alpha_2 + \alpha_3 = 0$

$\alpha(1,2,0) = \alpha_1 + 2\alpha_2 = 0$

so $\alpha_1 = -2\alpha_2$ $\quad \alpha_3 - 2\alpha_2 + \alpha_2 = 0$

So $\alpha_3 = \alpha_2$ $\therefore$ $(-2,1,1) \in \operatorname{ann} U$.

[4]

6. $4x_1^2 + 4x_1x_2 + x_2^2$

$= (2x_1 + x_2)^2$

$\therefore$ $\left.\begin{array}{l} \operatorname{Sig} = (1,0) \\ \operatorname{rank} = 1 \end{array}\right\}$ Sylvester's Thm

[4]

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(a) $\Leftarrow$ If $v_1 + \dots + v_k = 0 = 0 + \dots + 0$

since $v_i, 0 \in V_i$, defining property

of direct sum says $v_i = 0$ [2]

$\Rightarrow$ If $v_1 + \dots + v_k = w_1 + \dots + w_k$

with $v_i, w_i \in V_i$ then $v_i - w_i \in V_i$

& $\sum_{i=1}^{n} (v_i - w_i) = 0$ $\therefore$ $v_i - w_i = 0 \ \forall i$

i.e. $v_i = w_i \ \forall i$. [4]

(b) $u \in u + V_1 = w + V_2 \Rightarrow$

$u - w \in V_2$. Similarly $w - u \in V_1$

$\therefore$ $u - w \in V_1 \cap V_2$.

If $v_1 \in V_1$ then $u + v_1 = w + v_2$ some $v_2 \in V_2$

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$\therefore \; v_1 = v_2 + (w-u) \in V_2$

Similarly any $v_2 \in V_2$ has $v_2 \in V_1$

$\therefore \; V_1 \leq V_2$ & $V_2 \leq V_1 \;\therefore\; V_1 = V_2$.

[6]

(c) Let $\alpha(v) \neq 0$ then any $\lambda \in \mathbb{F}$

is $\frac{\lambda}{\alpha(v)} \alpha(v) = \alpha\left(\frac{\lambda}{\alpha(v)} v\right)$ so $\operatorname{im} \alpha = \mathbb{F}$

$\therefore$ First Iso$^{\underline{n}}$ Thm say $V/\ker \alpha \cong \operatorname{im} \alpha = \mathbb{F}$

In particular, $\dim V/\ker\alpha = \dim \mathbb{F} = 1$

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(a) Cauchy-Schwarz: $|\langle v,w\rangle|^2 \le \|v\|^2\|w\|^2$
with equality if $v = \lambda w$ some $\lambda \in \mathbb{F}$.

Let $v = (\sqrt{a}, \sqrt{b}, \sqrt{c}, \sqrt{d}, \sqrt{e})$ & $w = (1/\sqrt{a}, \dots, 1/\sqrt{e})$
& $\langle\, ,\rangle$ be dot product.

Then

$v \cdot w = \sqrt{a}/\sqrt{a} + \dots + \sqrt{e}/\sqrt{e} = 5$

$\|v\|^2 = a + b + \dots + e \qquad \|w\|^2 = 1/a + 1/b + \dots + 1/e$ [4]

whence result.

If $v = \lambda w$ then $\lambda = a = b = \dots = e$

so equality iff $a = b = \dots = e$. [1]

(b) Do Gram-Schmidt on cols of A:

set $v_1 = (1, -1, 0)$
$v_2 = (1, 0, -1)$
$v_3 = (3, 1, -1)$

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Then $w_1 = v_1$ & $\|w_1\|^2 = 2$ & $\boxed{u_1 = \frac{1}{\sqrt{2}}(1,-1,0)}$

$\langle w_1, v_2\rangle = 1$ ∴

$$w_2 = v_2 - \frac{\langle w_1, v_2\rangle w_1}{\|w_1\|^2} = (1,0,-1) - \frac{(1,-1,0)}{2} = \left(\tfrac{1}{2}, \tfrac{1}{2}, -1\right)$$

$\|w_2\|^2 = 3/2$ ∴ $\boxed{u_2 = \sqrt{2/3}\left(\tfrac{1}{2}, \tfrac{1}{2}, -1\right)}$

$\langle w_1, v_3\rangle = 3-1 = 2$ $\quad \langle w_2, v_3\rangle = 3/2 + 1/2 + 1 = 3$

∴ $$w_3 = v_3 - \sum_i^2 \frac{\langle w_i, v_3\rangle w_i}{\|w_i\|^2} =$$

$$(3,1,-1) - \frac{2}{2}(1,-1,0) - 2\left(\tfrac{1}{2}, \tfrac{1}{2}, -1\right) = (1,1,1)$$

∴ $\|w_3\|^2 = 3$ & $\boxed{u_3 = \frac{1}{\sqrt{3}}(1,1,1)}$

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$\therefore \quad Q = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ 0 & -2/\sqrt{6} & 1/\sqrt{3} \end{pmatrix} = \frac{1}{\sqrt{6}} \begin{pmatrix} \sqrt{3} & 1 & \sqrt{2} \\ -\sqrt{3} & 1 & \sqrt{2} \\ 0 & -2 & \sqrt{2} \end{pmatrix}$

$\therefore \quad R = Q^T A = \frac{1}{\sqrt{6}} \begin{pmatrix} \sqrt{3} & -\sqrt{3} & 0 \\ 1 & 1 & -2 \\ \sqrt{2} & \sqrt{2} & \sqrt{2} \end{pmatrix} \begin{pmatrix} 1 & 1 & 3 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix}$

$= \frac{1}{\sqrt{6}} \begin{pmatrix} 2\sqrt{3} & \sqrt{3} & 2\sqrt{3} \\ 0 & 3 & 6 \\ 0 & 0 & 3\sqrt{2} \end{pmatrix}$

[7]

(c) Nearest point is $\pi_U(v)$ for $v = (2,1,2)$

$\pi_U(v) = \langle u_1, v\rangle u_1 + \langle u_2, v\rangle u_2$ for u_1, u_2 o.n. basis of U.

But (b) says we can take

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$$u_1 = \frac{1}{\sqrt{2}}(1,-1,0) \qquad u_2 = \sqrt{2/3}\,(1/2, 1/2, -1)$$

$$\therefore \Pi_U(v) = \frac{1}{2}(1,-1,0) + \frac{2}{3}\left(-\frac{1}{2}\right)(1/2, 1/2, -1)$$

$$= \left(\frac{1}{2}-\frac{1}{6}, -\frac{1}{2}-\frac{1}{6}, \frac{1}{3}\right) = \boxed{\left(\frac{1}{3}, -\frac{2}{3}, \frac{1}{3}\right)}$$

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(a) $B(e_1, e_1) = -3 \neq 0$

so take $v_1 = e_1 = (1, 0, 0)$.

Now seek $v_2 = y$ s.t. $B(v_1, y) = 0$ i.e.

$$(1\ 0\ 0)\begin{pmatrix} -3 & 2 & -2 \\ 2 & -1 & 1 \\ -2 & 1 & -1 \end{pmatrix} y = (-3\ 2\ -2)y$$

$$= -3y_1 + 2y_2 - 2y_3 = 0$$

$v_2 = (0, 1, 1)$ will do or

$$(0, 1, 1)\begin{pmatrix} -3 & 2 & -2 \\ 2 & -1 & 1 \\ -2 & 1 & -1 \end{pmatrix} = (0\ 0\ 0)$$

So v_2 is in rad B.

Take another sol^{n} eg $v_3 = (2, 3, 0)$

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to get diagonalising basis.

$$B(v_3, v_3) = (2\ 3\ 0)\begin{pmatrix} -3 & 2 & -2 \\ 2 & -1 & 1 \\ -2 & 1 & -1 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}$$

$$= (2\ 3\ 0)\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = 3$$

[10]

(b) Sylvester says $s_B = (1,1)$ & and rank $= 2$ [4]

(c) $v_2 \in \operatorname{rad} B$ & $\dim \operatorname{rad} B = 1$

$\therefore \operatorname{rad} B = \operatorname{span}\{(0,1,1)\}$. [4]

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