Section A

1. Let V be a vector space over a field $\mathbb{F}$ and $U, W \leq V$ vector subspaces.

   What does it mean to say that V **is the internal direct sum of** U **and** W, that is, $V = U \oplus W$?

   What is a **complement** to U in V? [4]

2. Let V be an n-dimensional vector space over a field $\mathbb{F}$ and $U, W \leq V$ with $\dim U = 1$ and $\dim W = n - 1$.

   Show that either $U \leq W$ or $V = U \oplus W$. [4]

3. Let V be a finite-dimensional inner product space.

   What is an **orthonormal basis** of V?

   Let $u_1, \ldots, u_n$ be an orthonormal basis of V and $v, w \in V$. Prove that

$$v = \sum_{i=1}^{n} \langle u_i, v \rangle u_i$$

   and so deduce that

$$\langle v, w \rangle = \sum_{i=1}^{n} \langle v, u_i \rangle \langle u_i, w \rangle.$$

   [4]

4. Let V be a finite-dimensional complex inner product space and $\phi : V \to V$ a linear operator.

What is an **adjoint** of ϕ?

What does it mean to say that ϕ is **normal**?

Show that ϕ is normal if and only if, for all $v, w \in V$,

$$\langle \phi(v), \phi(w) \rangle = \langle \phi^*(v), \phi^*(w) \rangle.$$

[4]

5. Let V be a vector space over a field $\mathbb{F}$.

What is the **dual space** V^*?

Let $v_1, \ldots, v_n$ be a basis of V. Define the **dual basis** $v_1^*, \ldots, v_n^*$ of V^* to $v_1, \ldots, v_n$.

Prove that, for $\alpha \in V^*$,

$$\alpha = \sum_{i=1}^{n} \alpha(v_i) v_i^*.$$

[4]

6. What is the rank and signature of the quadratic form $Q : \mathbb{R}^2 \to \mathbb{R}$ given by

$$Q(x) = x_1^2 + x_1 x_2 + x_2^2.$$

[4]

Section B

7. (a) State the **First Isomorphism Theorem**. [4]

   (b) Let $\phi : V \to W$ be a linear surjection of vector spaces over a field $\mathbb{F}$ and $U = \ker \phi \leq V$.

   Prove that $V/U \cong W$. [4]

   (c) Let V be a vector space over a field $\mathbb{F}$ and $U \leq V$ a subspace such that V/U is finite-dimensional.

   Show that U has a complement W in V with $\dim W = \dim V/U$.

   Hint: For $q : V \to V/U$ the quotient map, let $q(v_1), \ldots, q(v_n)$ be a basis of V/U and consider the span of $v_1, \ldots, v_n$. [10]

8. (a) State and prove the Riesz Representation Theorem. [6]

   (b) Let U be the subspace of $\mathbb{R}^4$ spanned by $(1, -1, 0, 0)$, $(0, 1, -1, 0)$ and $(0, 0, 1, -1)$.

   (i) Find an orthonormal basis of U. [6]

   (ii) Stating any results from lectures that you use, find $u \in U$ such that $\|u - (1, 1, 2, 2)\|$ is as small as possible. [6]

9. Let A be the matrix

$$\begin{pmatrix} 1 & 2 & 0 \\ 2 & 0 & -1 \\ 0 & -1 & 2 \end{pmatrix}.$$

   (a) Find a matrix P such that $P^T AP$ is diagonal. [9]

   (b) Let V be a real vector space.

   What is a **symmetric bilinear form** on V?

   What are the **radical**, **rank** and **signature** of a symmetric bilinear form on V? [5]

   (c) Compute the rank and signature of the symmetric bilinear form B on $\mathbb{R}^3$ given by

$$B(x, y) = \mathbf{x}^T A \mathbf{y}.$$

[4]