University of Bath

DEPARTMENT OF MATHEMATICAL SCIENCES EXAMINATION

MA20216: Algebra 2A

MOCK EXAM

Answer ALL questions from Section A and TWO questions from Section B in the WHITE booklet.

For Section B, only your best two answers will contribute towards the assessment.

Use the YELLOW booklet for all your rough work. This booklet will not be collected.

No calculators may be brought in and used.

Section A

1. Let V be a vector space of a field $\mathbb{F}$.

   (a) Let U, W be subspaces of V. What does it mean to say that V is the *internal direct sum* of U and W: $V = U \oplus W$?

   (b) Let $\pi \in L(V)$ be a linear operator on V such that $\pi \circ \pi = \pi$.
   Show that $V = \operatorname{im} \pi \oplus \ker \pi$.

   [4]

2. State and prove the Cauchy–Schwartz inequality for an inner product space. [4]

3. Let V be an inner product space. What does it mean to say that a list of vectors $v_1, \ldots, v_k \in V$ are *orthonormal*?

   Prove that an orthonormal list of vectors is linearly independent. [4]

4. Let V be a complex inner product space.

   What is a *normal operator on* V?

   What is a *unitary transformation of* V?

   Show that a unitary operator is normal.

   Show that the eigenvalues of a unitary operator are of unit length. [4]

5. Let V be a vector space over a field $\mathbb{F}$, $U \leq V$ and $E \leq V^*$.

   What is the *solution set* $\operatorname{sol} E$ *of* E?

   What is the *annihilator* $\operatorname{ann} U$ *of* U?

   Prove that $U \leq \operatorname{sol} E$ if and only if $E \leq \operatorname{ann} U$. [4]

6. What is a *quadratic form* on a vector space over a field $\mathbb{F}$?
Define a quadratic form on $\mathbb{R}^2$ by $Q(x) = x_1 x_2$.
What is the rank and signature of Q? [4]

Section B

7. (a) Let V be a vector space over a field $\mathbb{F}$ and $U \leq V$.
What is a *coset* of U?
Suppose that $W \leq V$ also and that there are $x, y \in V$ such that

$$x + U = y + W.$$

Show that $U = W$. [6]

(b) State and prove the first isomorphism theorem. [6]

(c) Let V be a vector space over a field $\mathbb{F}$.

(i) Let $\alpha \in V^*$ with $\alpha \neq 0$. Show that $\dim(V/\ker\alpha) = 1$. [3]

(ii) Let $U \leq V$ with $\dim(V/U) = 1$.
Show that there is $\alpha \in V^*$ with $\ker\alpha = U$. [3]

8. (a) Let $x_1, \ldots, x_n \in \mathbb{R}$. Prove that

$$(x_1 + \cdots + x_n)^2 \leq n(x_1^2 + \cdots + x_n^2).$$

[6]

(b) What is an *orthogonal matrix*?
Prove that a square matrix is orthogonal if and only if its columns are orthonormal with respect to dot product. [4]

(c) Find the QR decomposition of the following matrix:

$$\begin{pmatrix} 0 & 0 & -2 \\ 2 & 1 & 1 \\ 0 & 2 & 1 \end{pmatrix}.$$

[8]

9. (a) State the Sufficiency Principle for a vector space V and prove it for the case of finite-dimensional V. [4]

(b) Let V be the vector space of complex polynomials of degree no more than $n-1$. Show there are complex numbers $a_1, \ldots, a_n$ such that, for all $p \in V$,

$$p(0) = \sum_{j=1}^{n} a_j p(e^{2\pi \mathbf{i} j/n}),$$

where $\mathbf{i} = \sqrt{-1}$. [6]

(c) Diagonalise the quadratic form $Q : \mathbb{R}^3 \to \mathbb{R}$ given by

$$Q(x) = x_1^2 + 2x_2^2 - 2x_3^2 - x_1x_2 + 2x_1x_3.$$

What is the signature of Q? [8]