Section A

1. Define $U, W \leq \mathbb{R}^4$ by $U = \operatorname{span}\{(1,2,2,1),(1,3,1,3)\}$, $W = \operatorname{span}\{(1,2,3,4)\}$.
Is it true that $U \oplus W = \mathbb{R}^4$? Justify your answer. [4]

2. Let V be a vector space, $U \leq V$ and $q : V \to V/U$ the quotient map.
Under what condition on U is q an isomorphism? [4]

3. Let $p = x^{17} + 5x + 1 \in \mathbb{R}[x]$, $v = (1,0,0) \in \mathbb{R}^3$ and $\phi = \phi_A \in L(\mathbb{R}^3)$, where

$$A = \begin{pmatrix} 1 & 5 & 7 \\ 0 & 2 & 9 \\ 0 & 0 & 6 \end{pmatrix}.$$

Compute $p(\phi)(v)$. [4]

4. Let $\phi \in L(V)$ be a linear operator on a finite-dimensional vector space V and suppose that

$$\Delta_\phi = (x-1)^3(x-17)^2, \qquad m_\phi = (x-1)(x-17)^2.$$

What is the Jordan normal form of ϕ? [4]

5. What is the *dual space* V^* of a vector space V over a field $\mathbb{F}$?
Define $\alpha, \beta \in (\mathbb{R}^3)^*$ by

$$\begin{aligned} \alpha(x) &= x_1 + 2x_2 - x_3, \\ \beta(x) &= 3x_1 - 3x_2. \end{aligned}$$

Write down a basis for $\operatorname{sol} E$ where $E = \operatorname{span}\{\alpha, \beta\}$. [4]

6. For which $t \in \mathbb{R}$ does the quadratic form $q_t : \mathbb{R}^2 \to \mathbb{R}$ given by

$$q_t(x) = x_1^2 + 2tx_1x_2 - 7x_2^2$$

have signature $(1, 1)$? [4]

Section B

7. Let $\phi : V \to W$ be a linear map of vector spaces and $A \leq W$. Define $\phi^{-1}(A)$ by

$$\phi^{-1}(A) = \{v \in V \mid \phi(v) \in A\}.$$

(a) Show that $\ker \phi \leq \phi^{-1}(A) \leq V$. [6]

(b) Let $U \leq V$ and $q : V \to V/U$ be the quotient map.

(i) Let $U \leq B \leq V$. Show that there is a subspace $A \leq V/U$ such that $B = q^{-1}(A)$. [6]

(ii) Let $A_1, A_2 \leq V/U$ and suppose that $q^{-1}(A_1) = q^{-1}(A_2)$. Prove that $A_1 = A_2$. [6]

8. (a) Let $\phi \in L(\mathbb{C})$ be a linear operator on a finite-dimensional complex vector space.

(i) What is the *minimum polynomial of* ϕ?

(ii) Show that the roots of the minimum polynomial are precisely the eigenvalues of ϕ.
(You may assume the Cayley–Hamilton theorem without proof.)

[9]

(b) Let $\phi = \phi_A \in L(\mathbb{C}^3)$ where

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ -2 & -3 & 0 \end{pmatrix}.$$

(i) What is the minimum polynomial of A?

(ii) What is the Jordan normal form of A?

[9]

9. (a) Which of the following are possible signatures of a quadratic form $q : \mathbb{R}^4 \to \mathbb{R}$?

(i) $(3, 0)$.

(ii) $(4, 1)$.

(iii) $(2, -2)$.

In each case, briefly justify your answer. [6]

(b) Find an invertible matrix P such that $P^T AP$ is diagonal where

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 0 & -1 \\ 1 & -1 & 1 \end{pmatrix}.$$

[12]