DEPARTMENT OF MATHEMATICAL SCIENCES
Outline solution to Examination Questions

UNIVERSITY OF BATH

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1/ No: $\dim U = 2$

$\dim W = 1$

$\therefore \dim U+W = \dim U + \dim W - \dim U \cap W$

≤ 3

while $\mathbb{R}^4$ has dim 4.

$\therefore \mathbb{R}^4 \neq U+W$ whether or not the sum is direct.

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2/ q always surjective

so bijective $\iff$ injective $\iff$

$\ker q = \{0\}$ but $\ker q = U$

$\therefore$ iso$^{\underline{m}}$ iff $U = \{0\}$

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3/ Observe that $\phi(v) = v$

i.e. eigenvector with eigenvalue $\underline{1}$

$$\therefore \quad p(\phi)v = p(1)v$$
$$= (1^{17} + 6\cdot 1 + 1)v$$
$$= 7v.$$

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4/ $\deg \Delta_\phi = 5 \therefore \dim V = 5$

Eigenvalues $1, 17$

$\dim G_\phi(1) = 3$

$\dim G_\phi(17) = 2$

Largest size of $J(1)$ is 1

Largest size of $J(17) = 2$

$\therefore$ only poss is

$$J_1(1) \oplus J_1(1) \oplus J_1(1) + J_2(17)$$

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$$V^* = L(V, \mathbb{F}) = \{\alpha : V \to \mathbb{F} \mid \alpha \text{ linear}\}$$

$$\text{Sol } E = \ker \alpha \cap \ker \beta$$

$$= \left\{ x \in \mathbb{R}^3 \;\middle|\; \begin{array}{l} x_1 + 2x_2 - x_3 = 0 \\ 3x_1 - 3x_2 = 0 \end{array} \right\}$$

$\therefore$ $x \in \text{sol} E$ iff $x_1 = x_2$

& then $3x_1 = x_3$

$\therefore$ $\text{sol} E = \text{span}\{(1,1,3)\}$.

& $(1,1,3)$ is a basis

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6/ We complete the square:

$$x_1^2 + 2t x_1 x_2 - 7x_2^2 =$$

$$(x_1 + t x_2)^2 - t^2 x_2^2 - 7x_2^2 =$$

$$(x_1 + t x_2)^2 - (t^2+7)x_2^2$$

& • $x_1 + t x_2, x_2$ lin. indep

• $(t^2+7) \geq 0$ $\therefore$

q_t has sig $(1,1)$ $\forall t$.

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7 (a) If $v \in \ker\varphi$ then $\varphi(v) = 0 \in A$

$\therefore v \in \varphi^{-1}(A) \therefore \ker\varphi \subseteq \varphi^{-1}(A)$.

If $v, w \in \varphi^{-1}(A)$ $\lambda \in \mathbb{F}$ then

$\varphi(v), \varphi(w) \in A$ so $\varphi(v+\lambda w) = \varphi(v) + \lambda\varphi(w) \in A$

$\therefore \varphi^{-1}(A) \leq V$.

(b)(i) Let $A = q(B) \leq V/U$

Then $v \in B \Rightarrow q(v) \in q(B) = A$ so

$v \in \varphi^{-1}(A) \therefore B \leq \varphi^{-1}(A)$

Let $v \in \varphi^{-1}(A) \therefore q(v) = q(u)$ some $b \in B$

$\therefore q(v-b) = 0$ i.e. $v - b \in \ker q = U$

But $U \leq B \therefore v \in B \therefore \varphi^{-1}(A) = B$

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(ii) For $A \leq V/U$, $q(q^{-1}(A)) = A$:

~~since~~ $v \in q^{-1}(A) \Rightarrow q(v) \in A$ so

$$q(q^{-1}(A)) \subseteq A.$$

Converse: if $a \in A$, $a = q(v)$ some $v \in V$ & so $v \in q^{-1}(A)$ so $a \in q(q^{-1}(A))$

$\therefore$ $q^{-1}(A_1) = q^{-1}(A_2) \Rightarrow q(q^{-1}(A_1)) = q(q^{-1}(A_2))$

$\therefore$ $A_1 = A_2$.

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(a) (i) Min poly is monic $p \in \mathbb{C}[x]$ of lowest degree s.t. $p(\phi) = 0$

(ii) If v is eigenvector of ϕ with eigenvalue λ & m_ϕ is min. poly

then $0 = m_\phi(\phi)(v) = m_\phi(\lambda)v$

$\therefore \ m_\phi(\lambda) = 0$

$\therefore$ any eigenval is root of m_ϕ.

By Cayley-Hamilton, $m_\phi \mid \Delta_\phi$ $\therefore$ any root of m_ϕ is root of Δ_ϕ $\therefore$ an eigenval.

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(i) First compute Δ_A:

$$\begin{vmatrix} 2-x & 1 & 0 \\ 0 & 2-x & 0 \\ -2 & -3 & -x \end{vmatrix} = -(2-x)^2 x$$

$\therefore$ min poly either

$x(x-2)$ or $(x-2)^2 x$.

Try $x(x-2)$:

$$\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ -2 & -3 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ -2 & -3 & -2 \end{pmatrix} = \begin{pmatrix} 0 & 2 & \\ 0 & \cdots & \\ 0 & & \end{pmatrix} \neq 0$$

$\therefore$ $M_A = (x-2)^2 x$

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(ii) From Δ_φ we see that

$\dim G_\varphi(0) = 1$ or from m_φ largest

$\dim G_\varphi(2) = 2$ Jordan block in

$G_\varphi(0) = J_1(0)$ or in $G_\varphi(2)$ is $J_2(2)$

$\therefore$ JNF: $J_1(0) + J_2(2)$

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(a) (i) Yes: eg $x_1^2+x_2^2+x_3^2$

(ii) No: $(4,1)$ gives rank $=5>\dim \mathbb{R}^4$

(iii) No: sig is (p,q) with $p,q\geq 0$.

(b) Find diagonalising basis of $B=B_A$

$A_{11}\neq 0$ so ~~eg~~ take $v_1=e_1$.

Now want y s.t $B(e_1,y)=0$ i.e.

$$(1,0,0)\begin{pmatrix}1&2&1\\2&0&-1\\1&-1&1\end{pmatrix}\underline{y}=0$$

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i.e. $(1\ 2\ 1)\underline{y} = y_1 + 2y_2 + y_3 = 0$

Try $v_2 = (2, -1, 0)$

$$(2\ -1\ 0)\begin{pmatrix} 1 & 2 & 1 \\ 2 & 0 & -1 \\ 1 & -1 & 1 \end{pmatrix} = (0\ 4\ 3)$$

$$\therefore\ B(v_2, v_2) = (0\ 4\ 3)\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} = -4 \neq 0\ ✓$$

Now need $y = v_3$ with $B(v_1, y) = B(v_2, y) = 0$

i.e. $y_1 + 2y_2 + y_3 = 0$

$4y_2 + 3y_3 = 0$

so take $y_2 = 3$ $y_3 = -4$ so that $y_1 = -2$

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✓ $v_3 = (-2, 3, -4$

so $$P = \begin{pmatrix} 1 & 2 & -2 \\ 0 & -1 & 3 \\ 0 & 0 & -4 \end{pmatrix}$$

✓ $$P^T A P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 36 \end{pmatrix}$$

Total