Section A

1. Let $U, W \le \mathbb{R}^3$ be given by $U = \operatorname{span}\{(1,1,1),(1,2,1)\}$, $W = \operatorname{span}\{(1,0,0),(0,0,1)\}$.
   Write down bases for $U \cap W$ and $U + W$. [4]

2. Let $\phi \in L(V)$ be a linear operator on a finite-dimensional vector space V such that
   $$\ker \phi \cap \operatorname{im} \phi = \{0\}.$$
   Prove that $\ker \phi \oplus \operatorname{im} \phi = V$. [4]

3. Find the minimal polynomial of $A \in M_3(\mathbb{R})$ given by
   $$A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 2 \\ 1 & 0 & 0 \end{pmatrix}.$$
   [4]

4. Let $\phi \in L(V)$ be a linear operator on a complex vector space V with characteristic polynomial $(x-3)^4(x-2)^2$ and minimal polynomial $(x-3)^2(x-2)$.
   What are the possible Jordan normal forms of ϕ? [4]

5. Let $E \le (\mathbb{R}^3)^*$ be spanned by α given by
   $$\alpha(x) = x_1 + 2x_2 - x_3.$$
   Write down a basis of $\operatorname{sol} E$. [4]

6. Define the *rank* and *signature* of a symmetric bilinear form B on a real, finite-dimensional vector space.
   State Sylvester's Law of Inertia. [4]

Section B

7. Let V be a vector space over a field $\mathbb{F}$ and $\phi \in L(V)$ a linear operator on V.

(a) Suppose that $V = V_1 \oplus V_2$ with each V_i ϕ-invariant. Show that

$$\ker \phi = \ker \phi_{|V_1} \oplus \ker \phi_{|V_2}.$$

[6]

(b) Let $U \leq V$ be ϕ-invariant and let $q : V \to V/U$ be the quotient map.

(i) Show that there is a well-defined linear operator $\bar{\phi}$ on V/U such that

$$\bar{\phi}(q(v)) = q(\phi(v)),$$

for all $v \in V$.

(ii) If $\dim V = n$, $u_1, \ldots, u_k$ is a basis of U and $q(v_1), \ldots, q(v_{n-k})$ is a basis of V/U, show that $u_1, \ldots, u_k, v_1, \ldots, v_{n-k}$ is a basis of V.

[12]

8. Let A be given by

$$A = \begin{pmatrix} 1 & -1 & 2 \\ -3 & -1 & -3 \\ 1 & 1 & 0 \end{pmatrix}.$$

The characteristic polynomial of A is $(2-x)(x+1)^2$ (you do not need to prove this).

(a) Compute the minimum polynomial of A. [5]

(b) Find the Jordan normal form of A. [4]

(c) Find a Jordan basis for A. [9]

9. (a) Let $q : \mathbb{R}^5 \to \mathbb{R}$ be a quadratic form. Which of the following are possible signatures of q?

(i) $(4, 2)$.

(ii) $(2, -1)$.

(iii) $(2, 1)$.

(iv) $(2, 3)$.

In each case, briefly justify your answer. [8]

(b) Let $t \in \mathbb{R}$ and define A_t by

$$A_t = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & t \\ -1 & t & 0 \end{pmatrix}.$$

Find an invertible matrix P_t such that $P_t^T A_t P_t$ is diagonal. [10]