

### Section A

1. Define subspaces  $U, W \leq \mathbb{R}^3$  by  $U = \text{span}\{(1, 0, -1), (2, -1, 0)\}$  and  $W = \text{span}\{(1, 1, 1)\}$ . Show that the sum  $U + W$  is direct. [4]
  
2. With  $U, W$  as in question 1, find  $u \in U$  and  $w \in W$  such that
$$(1, 2, 3) = u + w.$$
[4]
  
3. Use dot product to make  $\mathbb{R}^3$  into an inner product space. With  $W$  as in question 1, find a basis for  $W^\perp$ . [4]
  
4. Let  $V$  be a complex inner product space and  $\phi$  a unitary operator on  $V$ . If  $\lambda$  is an eigenvalue of  $\phi$ , show that  $\|\lambda\| = 1$ . [4]
  
5. Let  $V$  be a vector space over a field  $\mathbb{F}$  and  $v, w \in V$  with  $v \neq w$ . Show that there is  $\alpha \in V^*$  such that  $\alpha(v) \neq \alpha(w)$ . [4]
  
6. For  $t \in \mathbb{R}$ , define a quadratic form  $Q_t$  on  $\mathbb{R}^2$  by  $Q_t(x) = x_1^2 - 2x_1x_2 + tx_2^2$ . For which  $t$  does  $Q_t$  have rank 1? [4]

## Section B

7. Let  $V$  be a vector space over a field  $\mathbb{F}$ .

(a) Let  $\pi_1, \pi_2, \pi_3 \in L(V)$  satisfy

$$\begin{aligned}\pi_i \circ \pi_j &= \delta_{ij} \pi_i, \quad \text{for } 1 \leq i, j \leq 3 \\ \text{id}_V &= \pi_1 + \pi_2 + \pi_3.\end{aligned}$$

(i) Show that  $V = \text{im } \pi_1 \oplus \text{im } \pi_2 \oplus \text{im } \pi_3$ .

(ii) Show that  $\ker \pi_1 = \text{im } \pi_2 \oplus \text{im } \pi_3$ .

[12]

(b) Let  $\phi \in L(V)$  be a linear operator on  $V$  and  $U \leq V$  a  $\phi$ -invariant subspace. Let  $q : V \rightarrow V/U$  be the quotient map.

Show that there is a well-defined linear operator  $\tilde{\phi} \in L(V/U)$  such that

$$\tilde{\phi}(q(v)) = q(\phi(v)),$$

for  $v \in V$ .

[6]

8. (a) Let  $V$  be a finite-dimensional complex inner product space and  $\phi \in L(V)$  a linear operator.

Show that  $\lambda$  is an eigenvalue of  $\phi$  if and only if  $\bar{\lambda}$  is an eigenvalue of  $\phi^*$ . [6]

(b) Let  $U = \text{span}\{(1, 1, -1, -1), (1, 0, 0, -1)\} \leq \mathbb{R}^4$  and view  $\mathbb{R}^4$  as an inner product space using dot product.

(i) Compute  $U^\perp$ .

(ii) Find an orthonormal basis of  $U^\perp$ .

(iii) Compute the orthogonal projections of  $(1, 2, 3, 1)$  onto  $U$  and  $U^\perp$ .

[12]

9. (a) Let  $V$  be a finite-dimensional real vector space and  $\alpha, \beta \in V^*$  linearly independent linear functionals.

Define a symmetric bilinear form  $B$  on  $V$  by

$$B(v, w) = \frac{1}{2}(\alpha(v)\beta(w) + \alpha(w)\beta(v)).$$

Compute the rank and signature of  $B$ . [6]

- (b) Let  $t \in \mathbb{R}$  and define  $A_t$  by

$$A_t = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 0 & t \\ 0 & t & 3 \end{pmatrix}.$$

For which  $t$  does the symmetric bilinear form  $B_{A_t}$  have signature  $(3, 0)$ ? [12]