Section A

1. Define subspaces $U, W \leq \mathbb{R}^3$ by $U = \text{span}\{(1, 0, \text{-}1), (2, \text{-}1, 0)\}$ and $W = \text{span}\{(1, 1, 1)\}$.

   Show that the sum $U + W$ is direct. [4]

2. With U, W as in question 1, find $u \in U$ and $w \in W$ such that

$$(1, 2, 3) = u + w.$$

   [4]

3. Use dot product to make $\mathbb{R}^3$ into an inner product space.

   With W as in question 1, find a basis for $W^{\perp}$. [4]

4. Let V be a complex inner product space and ϕ a unitary operator on V.

   If λ is an eigenvalue of ϕ, show that $\|\lambda\| = 1$. [4]

5. Let V be a vector space over a field $\mathbb{F}$ and $v, w \in V$ with $v \neq w$. Show that there is $\alpha \in V^*$ such that $\alpha(v) \neq \alpha(w)$. [4]

6. For $t \in \mathbb{R}$, define a quadratic form Q_t on $\mathbb{R}^2$ by $Q_t(x) = x_1^2 - 2x_1x_2 + tx_2^2$.

   For which t does Q_t have rank 1? [4]

Section B

7. Let V be a vector space over a field $\mathbb{F}$.

(a) Let $\pi_1, \pi_2, \pi_3 \in L(V)$ satisfy

$$\pi_i \circ \pi_j = \delta_{ij}\pi_i, \quad \text{for } 1 \leq i, j \leq 3$$
$$\mathrm{id}_V = \pi_1 + \pi_2 + \pi_3.$$

(i) Show that $V = \operatorname{im}\pi_1 \oplus \operatorname{im}\pi_2 \oplus \operatorname{im}\pi_3$.

(ii) Show that $\ker\pi_1 = \operatorname{im}\pi_2 \oplus \operatorname{im}\pi_3$.

[12]

(b) Let $\phi \in L(V)$ be a linear operator on V and $U \leq V$ a ϕ-invariant subspace. Let $q : V \to V/U$ be the quotient map.

Show that there is a well-defined linear operator $\tilde{\phi} \in L(V/U)$ such that

$$\tilde{\phi}(q(v)) = q(\phi(v)),$$

for $v \in V$. [6]

8. (a) Let V be a finite-dimensional complex inner product space and $\phi \in L(V)$ a linear operator.

Show that λ is an eigenvalue of ϕ if and only if $\bar{\lambda}$ is an eigenvalue of ϕ^*. [6]

(b) Let $U = \operatorname{span}\{(1, 1, \text{-}1, \text{-}1), (1, 0, 0, \text{-}1)\} \leq \mathbb{R}^4$ and view $\mathbb{R}^4$ as an inner product space using dot product.

(i) Compute $U^\perp$.

(ii) Find an orthonormal basis of $U^\perp$.

(iii) Compute the orthogonal projections of $(1, 2, 3, 1)$ onto U and $U^\perp$.

[12]

9. (a) Let V be a finite-dimensional real vector space and $\alpha, \beta \in V^*$ linearly independent linear functionals.

Define a symmetric bilinear form B on V by

$$B(v, w) = \frac{1}{2}(\alpha(v)\beta(w) + \alpha(w)\beta(v)).$$

Compute the rank and signature of B. [6]

(b) Let $t \in \mathbb{R}$ and define A_t by

$$A_t = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 0 & t \\ 0 & t & 3 \end{pmatrix}.$$

For which t does the symmetric bilinear form B_{A_t} have signature $(3, 0)$? [12]