DEPARTMENT OF MATHEMATICAL SCIENCES
Outline solution to Examination Questions

UNIVERSITY OF BATH

Unit Code MA20216	Unit Title Algebra 2A		
Academic Year 2019-20	Examiner Fran		
Semester 1	Question No. 1 — 6	Page 1	of 2

Part | Mark

1. $V = U \oplus W$ iff any $v \in V$ can be written $v = u + w$ for unique $u \in U, w \in W$. [2]

A complement to U in V is $W \leq V$ s.t. $V = U \oplus W$. [2]

2. $\dim U \cap W \leq \dim U = 1$ $\therefore$ either
- $\dim U = \dim U \cap W = 1 \Rightarrow U \cap W = U \Rightarrow U \leq W$ OR [1]
- $\dim U \cap W = 0 \Rightarrow U \cap W = \{0\}$ so that $U + W$ is direct sum.

Then $\dim U \oplus W = \dim U + \dim W = n = \dim V$ so $V = U \oplus W$. [3]

3. Orthonormal basis is basis $u_1, \dots, u_n$ s.t $\langle u_i, u_j \rangle = \delta_{ij}$ $1 \leq i, j \leq n$. [1]

$v \in V$ then $v = \sum_i \lambda_i u_i$ & $\langle u_j, v \rangle = \sum \lambda_i \langle u_j, u_i \rangle = \lambda_j$ $\therefore$

$$v = \sum_{i=1}^n \langle u_i, v \rangle u_i$$ [1]

Similarly $w = \sum_{i=1}^n \langle u_i, w \rangle u_i$ so

$$\langle v, w \rangle = \sum_{i,j} \overline{\langle u_i, v \rangle} \langle u_j, w \rangle \langle u_i, u_j \rangle = \sum_i \overline{\langle u_i, v \rangle} \langle u_i, w \rangle$$

$$= \sum_{i=1}^n \langle v, u_i \rangle \langle u_i, w \rangle$$ using sesquilinearity & conj. symmetry of $\langle , \rangle$. [2]

Total

Unit Code MA20216	Unit Title Algebra 2A	
Academic Year 2019-20	Examiner Fran	
Semester 1	Question No. 1—6	Page 2 of 2

Part | Mark

4. Adjoint is lin. op $\varphi^*: V \to V$ s.t. $\langle v, \varphi(w)\rangle = \langle \varphi^*(v), w\rangle \ \forall v,w \in V$. [1]

φ <u>normal</u> if $\varphi \circ \varphi^* = \varphi^* \circ \varphi$ [1]

$$\varphi \circ \varphi^* = \varphi^* \circ \varphi \underset{\text{non-deg. lemma}}{\iff} \langle \varphi\circ\varphi^*(v) - \varphi^*\circ\varphi(v), w\rangle = 0 \quad \forall v,w \in V$$

$$\iff \langle \varphi^*(v), \varphi^*(w)\rangle = \langle \varphi(v), (\varphi^*)^*(w)\rangle$$

$$\underset{(\varphi^*)^* = \varphi}{\iff} \langle \varphi^*(v), \varphi^*(w)\rangle = \langle \varphi(v), \varphi(w)\rangle \quad \forall v,w \in V.$$ [2]

5. Dual space: $V^* := \{ f : V \to \mathbb{F} \mid f \text{ linear}\}$ [1]

$v_i^* \in V^*$ defined by $v_i^*(v_j) = \delta_{ij}$ & extend by linearity. [1]

$$\cancel{\alpha(v_j)} \ \sum \alpha(v_i) v_i^*(v_j) = \sum \alpha(v_i)\delta_{ij} = \alpha(v_j)$$

$\therefore$ $\sum \alpha(v_i)v_i^*$ & α agree on basis $v_1, \dots, v_n$ & $\therefore$ everywhere. [2]

6. $x_1^2 + x_1x_2 + x_2^2 = (x_1 + \tfrac{1}{2}x_2)^2 + \tfrac{3}{4}x_2^2$

& $x_1 + \tfrac{1}{2}x_2$, x_2 lin. indep $\therefore$ sig $= (2,0)$ & rank $= 2$. [4]

Total

UNIVERSITY OF BATH

Unit Code MA20216	Unit Title Algebra 2A		
Academic Year 2019-20	Examiner Fran		
Semester 1	Question No. 7	Page 1	of 1

Part | Mark

(a) Let $\varphi: V \to W$ be linear. Then $V/\ker\varphi \cong \operatorname{im}\varphi$ via $\bar{\varphi}: V/\ker\varphi \to \operatorname{im}\varphi$ given by $\bar{\varphi}(q(v)) = \varphi(v)$ where $q: V \to V/\ker\varphi$ is quotient map. This is well-defined linear iso$^{\underline{m}}$. [4]

(b) $\operatorname{im}\varphi = W$ $\ker\varphi = U$ so First Iso$^{\underline{m}}$ Thm says $V/U \cong W$. [4]

(c) Let $q(v_1) \ldots q(v_n)$ be basis of V/U & set $W = \operatorname{span}\{v_1, \ldots, v_n\}$

- $v_1, \ldots, v_n$ lin. indep: $\sum \lambda_i v_i = 0 \Rightarrow \sum \lambda_i q(v_i) = 0 \Rightarrow \lambda_i = 0 \ \forall i$ since $q(v_1) \ldots q(v_n)$ lin. indep. $\therefore$ $\dim W = n = \dim V/U$. [3]

- $W \cap U = \{0\}$: if $\sum \lambda_i v_i = u \in U$ then $q(\sum \lambda_i v_i) = q(u) = 0$ $\therefore \sum \lambda_i q(v_i) = 0$ & once more $\lambda_i = 0 \ \forall i$ so $u = 0$. [3]

- $V = U + W$: Let $v \in V$ & write $q(v) = \sum \lambda_i q(v_i) = q(\sum \lambda_i v_i)$ then $q(v - \sum \lambda_i v_i) = 0$ i.e. $v - \sum \lambda_i v_i \in U$

$$\therefore \quad v = \underbrace{\sum \lambda_i v_i}_{\in W} + \underbrace{(v - \sum \lambda_i v_i)}_{\in U} \in U + W.$$ [4]

Total

Unit Code MA20216	Unit Title Algebra 2A	
Academic Year 2019-20	Examiner Fran	
Semester 1	Question No. 8	Page 1 of 2

Part | Mark

(a) Riesz: Let V be finite dim^{l} inner product space & $\alpha \in V^*$. Then there is $w \in V$ s.t. $\forall v \in V$

$$\alpha(v) = \langle w, v \rangle.$$

In fact, $w = \sum_{i=1}^{n} \overline{\alpha(u_i)}\, u_i$ for $u_1, \dots, u_n$ o.n. basis of V. [2]

Pf With w so defined

$$\langle w, u_j \rangle = \left\langle \sum \overline{\alpha(u_i)} u_i, u_j \right\rangle = \sum \alpha(u_i) \langle u_i, u_j \rangle = \alpha(u_j) \ \forall j$$

$\therefore$ if $v = \sum \lambda_i u_i$, $\langle w, v \rangle = \sum \lambda_i \langle w, u_i \rangle = \sum \lambda_i \alpha(u_i) = \alpha(v)$.

[4]

(b) (i) Set $v_1 = (1,-1,0,0)$, $v_2 = (0,1,-1,0)$ $v_3 = (0,0,1,-1)$

$w_1 := v_1$ $\quad u_1 = w_1/\|w_1\| = 1/\sqrt{2}\,(1,-1,0,0)$ $\quad \langle w_1, v_2 \rangle = -1$

$$w_2 = v_2 - \frac{\langle w_1, v_2 \rangle w_1}{\|w_1\|^2} = (0,1,-1,0) + \tfrac{1}{2}(1,-1,0,0) = (\tfrac12, \tfrac12, -1, 0)$$

$\|w_2\|^2 = 3/2 \quad u_2 = \sqrt{2/3}\,(\tfrac12, \tfrac12, -1, 0)$

$\langle w_1, v_3 \rangle = 0 \quad \langle w_2, v_3 \rangle = -1$

$$w_3 = v_3 - \cancel{\frac{\langle w_1, v_3 \rangle w_1}{\|w_1\|^2}} - \frac{\langle w_2, v_3 \rangle w_2}{\|w_2\|^2} = (0,0,1,-1) + \tfrac{2}{3}(\tfrac12, \tfrac12, -1, 0)$$

$$= (\tfrac13, \tfrac13, \tfrac13, -1)$$

Total

Unit Code MA20216	Unit Title Algebra 2A	
Academic Year 2019-20	Examiner Fran	
Semester 1	Question No. 8	Page 2 of 2

Part | Mark

& $\|w_3\|^2 = 1 + 3/9 = 4/3 \quad \therefore\ u_3 = \sqrt{3/4}\,(1/3, 1/3, 1/3, -1)$

$\therefore$ o.n. basis of U is $\frac{1}{\sqrt{2}}(1,-1,0,0),\ \sqrt{2/3}\,(1/2, 1/2, -1, 0),\ \sqrt{3/4}\,(1/3, 1/3, 1/3, -1)$

[6]

(ii) Lectures: u with smallest $\|\cancel{(1,1,2,2)}\ v - u\|$ is

$\pi_U(\cancel{(1,1,2,2)}\ v) = \sum_{i=1}^{k} \langle u_i, v\rangle u_i$ for $u_1 \dots u_k$ o.n. basis of U.

In case at hand, with $v = (1,1,2,2)$

$\pi_U(v) = \sum^3 \frac{\langle w_i, v\rangle w_i}{\|w_i\|^2}$.

Now $\langle w_1, v\rangle = 0 \quad \langle w_2, v\rangle = -1 \quad \langle w_3, v\rangle = -2/3$

$\therefore\ \pi_U(v) = -2/3\,(1/2, 1/2, -1, 0)\ \cancel{+}\ -2/3\ 3/4\,(1/3, 1/3, 1/3, -1)$

$= (-1/3, -1/3, 2/3, 0) - (1/6, 1/6, 1/6, -1/2)$

$= (-1/2, -1/2, 1/2, 1/2) = 1/2\,(-1, -1, 1, 1)$

[6]

Total

DEPARTMENT OF MATHEMATICAL SCIENCES
Outline solution to Examination Questions

UNIVERSITY OF BATH

Unit Code MA20216	Unit Title Algebra 2A		
Academic Year 2019-20	Examiner Fran		
Semester 1	Question No. 9	Page 1	of 1

Part | Mark

(a) P has cols given by a diagonalising basis of $B(x,y) = \underline{x}^T A \underline{y}$

Exploit 0 in $(1,3)$ slot to see that $v_1 = (1,0,0)$, $v_2 = (0,0,1)$ is start of such a basis. So seek $y = v_3$ s.t. $B(v_1,y) = B(v_2,y) = 0$ i.e.

$(1\ 0\ 0) A \underline{y} = (1, 2, 0)\underline{y} = y_1 + 2y_2 = 0$

$(0\ 0\ 1) A \underline{y} = (0\ {-1}\ 2)\underline{y} = -y_2 + 2y_3 = 0$ so $y = (-4, 2, 1)$ will do

$\therefore\ P = \begin{pmatrix} 1 & 0 & -4 \\ 0 & 0 & 2 \\ 0 & 1 & 1 \end{pmatrix}$. Check: $P^T A P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ -4 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 \\ 2 & 0 & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & -4 \\ 0 & 0 & 2 \\ 0 & 1 & 1 \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ -4 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 2 & -1 & -9 \\ 0 & 2 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -18 \end{pmatrix}$ ✓ [9]

(b) symmetric bilinear form is $B: V \times V \to \mathbb{F}$ s.t.
- $B(v, v_1 + \lambda v_2) = B(v, v_1) + \lambda B(v, v_2)$ $\forall v, w, v_1, v_2 \in V$ $\lambda \in \mathbb{F}$
- $B(v,w) = B(w,v)$ [1]

$\operatorname{rad} B := \{ v \in V \mid B(v,w) = 0\ \forall w \in V \}$ [1]

$\operatorname{rank} B = \dim V - \dim \operatorname{rad} B$ [1]

$\operatorname{sig} B = (p,q)$ where $p = \max\{ \dim U \mid U \leq V\ B|_{U \times U} \text{ pos. def} \}$

$q = \max\{ \dim W \mid W \leq V\ -B|_{W \times W} \text{ pos. def} \}$ [2]

(c) Sylvester says $p = \#$ pos entries of diag matrix rep. $A = 2$

$q = \#$ neg entries $= 1$ [4]

$\therefore \operatorname{rank} = p + q = 3$.

Total