Section A

1. Let V be a vector space over a field $\mathbb{F}$ and let $V_1, \dots, V_k \leq V$ be linear subspaces.
What is the *sum* $V_1 + \cdots + V_k$?
What does it mean to say that the sum $V_1 + \cdots + V_k$ is *direct*? [4]

2. Let V be a vector space over a field $\mathbb{F}$, $v \in V$ and $U \leq V$ a linear subspace.
What is the *coset* $v + U$?
Let $w \in V$. Show that $w \in v+U$ if and only if $\lambda v + (1-\lambda)w \in v+U$, for all $\lambda \in \mathbb{F}$. [4]

3. Let V be a vector space over $\mathbb{C}$.
What is an *inner product* on V?
For $x, y \in \mathbb{C}^2$, let
$$\langle x, y \rangle = \overline{x_1} y_2 + \overline{x_2} y_1.$$
Is $\langle \, , \, \rangle$ an inner product on $\mathbb{C}^2$? (You must justify your answer.) [4]

4. Let V be an inner product space over $\mathbb{C}$ and ϕ a linear operator on V.
What is an *adjoint* of ϕ?
What does it mean to say that ϕ is *normal*? [4]

5. Let V be a vector space over a field $\mathbb{F}$.
What is the *dual space* V^* of V?
Let $U \leq V$ be a linear subspace. What is the *annihilator* $\operatorname{ann} U$ of U? [4]

6. Compute the rank and signature of the quadratic form Q on $\mathbb{R}^2$ given by
$$Q(x) = x_1^2 - 6x_1x_2 + 9x_2^2?$$
[4]

Section B

7. Let V, W be finite-dimensional vector spaces over a field $\mathbb{F}$, $U \leq V$ a linear subspace and $\phi : U \to W$ be a linear map.

   (a) Prove that there is a linear map $\Phi : V \to W$ such that
   $$\Phi(u) = \phi(u),$$
   for all $u \in U$. [6]

   (b) Prove that the restriction map $r : L(V, W) \to L(U, W)$ given by $r(\Phi) = \Phi_{|U}$ is a linear surjection.
   What is its kernel? [6]

   (c) State the First Isomorphism Theorem. [2]

   (d) Prove that $U^* \cong V^* / \operatorname{ann} U$. [4]

8. (a) State and prove the Cauchy–Schwarz inequality. [6]

   (b) Let $a_1, \ldots, a_n \in \mathbb{R}$. Prove that
   $$\Big(\sum_{i=1}^{n} a_i/n\Big)^2 \leq \sum_{i=1}^{n} a_i^2/n.$$
   [3]

   (c) Compute the QR decomposition of the matrix A given by
   $$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -3 & 3 \end{pmatrix}.$$
   [9]

9. (a) Let V be a real vector space.
   What is a *bilinear form* on V?
   What is a *quadratic form* on V?
   What are the *rank* and *signature* of a quadratic form?
   State Sylvester's Law of Inertia. [9]

   (b) Diagonalise the quadratic form $Q : \mathbb{R}^3 \to \mathbb{R}$ given by
   $$Q(x) = x_1^2 + 3x_2^2 - x_3^2 + 2x_1x_2 + 4x_2x_3.$$
   What are the rank and signature of Q? [9]