Section A

1. Let V be a vector space over a field $\mathbb{F}$ and let $V_1, \ldots, V_k \leq V$ be linear subspaces.

   What is the **sum** $V_1 + \cdots + V_k$?

   What does it mean to say that the sum $V_1 + \cdots + V_k$ is **direct**? [4]

2. Let V be a vector space over a field $\mathbb{F}$, $v \in V$ and $U \leq V$ a linear subspace.

   What is the **coset** $v + U$?

   Let $w \in V$. Show that $w \in v + U$ if and only if $\lambda v + (1 - \lambda)w \in v + U$, for all $\lambda \in \mathbb{F}$. [4]

3. Let V be a vector space over $\mathbb{C}$.

   What is an **inner product** on V?

   For $x, y \in \mathbb{C}^2$, let

   $$\langle x, y \rangle = \overline{x_1} y_2 + \overline{x_2} y_1.$$

   Is $\langle\,,\,\rangle$ an inner product on $\mathbb{C}^2$? (You must justify your answer.) [4]

4. Let V be an inner product space over $\mathbb{C}$ and ϕ a linear operator on V.

   What is an **adjoint** of ϕ?

   What does it mean to say that ϕ is **normal**? [4]

5. Let V be a vector space over a field $\mathbb{F}$.

   What is the **dual space** V^* of V?

   Let $U \leq V$ be a linear subspace. What is the **annihilator** $\operatorname{ann} U$ of U? [4]

6. Compute the rank and signature of the quadratic form Q on $\mathbb{R}^2$ given by

$$Q(x) = x_1^2 - 6x_1x_2 + 9x_2^2?$$

[4]

Section B

7. Let V, W be finite-dimensional vector spaces over a field $\mathbb{F}$, $U \leq V$ a linear subspace and $\phi : U \to W$ be a linear map.

   (a) Prove that there is a linear map $\Phi : V \to W$ such that

   $$\Phi(u) = \phi(u),$$

   for all $u \in U$. [6]

   (b) Prove that the restriction map $r : L(V, W) \to L(U, W)$ given by $r(\Phi) = \Phi_{|U}$ is a linear surjection.

   What is its kernel? [6]

   (c) State the First Isomorphism Theorem. [2]

   (d) Prove that $U^* \cong V^* / \operatorname{ann} U$. [4]

8. (a) State and prove the Cauchy–Schwarz inequality. [6]

(b) Let $a_1, \dots, a_n \in \mathbb{R}$. Prove that

$$\Big(\sum_{i=1}^{n} a_i/n\Big)^2 \leq \sum_{i=1}^{n} a_i^2/n.$$

[3]

(c) Compute the QR decomposition of the matrix A given by

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -3 & 3 \end{pmatrix}.$$

[9]

9. (a) Let V be a real vector space.

What is a **bilinear form** on V?

What is a **quadratic form** on V?

What are the **rank** and **signature** of a quadratic form?

State Sylvester's Law of Inertia. [9]

(b) Diagonalise the quadratic form $Q : \mathbb{R}^3 \to \mathbb{R}$ given by

$$Q(x) = x_1^2 + 3x_2^2 - x_3^2 + 2x_1x_2 + 4x_2x_3.$$

What are the rank and signature of Q? [9]