Section A

1. Let V be a vector space over a field $\mathbb{F}$ and $U \leq V$ a subspace.

   What does it mean to say that $v, w \in V$ are *congruent modulo* U: $v \equiv w \mod U$?

   Show that congruence modulo U is an equivalence relation on V. [4]

2. Let V be an inner product space. What does it mean to say that $u_1, \ldots, u_n$ is an *orthonormal basis* of V?

   If $u_1, \ldots, u_n$ is an orthonormal basis of V and $v \in V$, show that

$$v = \sum_{i=1}^{n} \langle u_i, v \rangle u_i.$$

   [4]

3. Let V be an inner product space. State and prove the parallelogram identity for $v, w \in V$. [4]

4. Let ϕ be a linear operator on an inner product space V.

   What is an *adjoint* of ϕ?

   Let ϕ be self-adjoint. Show:

   (a) any eigenvalue of ϕ is real;

   (b) if v, w are eigenvectors of ϕ with distinct eigenvalues, then $\langle v, w \rangle = 0$.

   [4]

5. Let V be a vector space over a field $\mathbb{F}$.

   What is the *dual space* V^* of V?

   Let $\phi : V \to W$ be a linear map. Define the *transpose of* ϕ.

   If $\psi : W \to U$ is also linear, show that $(\psi \circ \phi)^T = \phi^T \circ \psi^T$. [4]

6. What is the rank and signature of the quadratic form Q on $\mathbb{R}^2$ given by

$$Q(x) = x_1^2 + 2x_1x_2 - x_2^2?$$

[4]

Section B

7. (a) Let V be a vector space over a field $\mathbb{F}$ and $U \leq V$ a subspace.

    (i) What is a *complement to U in V*? [2]

    (ii) Suppose that V is finite-dimensional. Show that U has a complement in V. [4]

    (iii) Let W be a complement to U and let V/U be the quotient space. Show that $W \cong V/U$. [4]

   (b) Let $\phi : V \to W$ be a surjective linear map of finite-dimensional vector spaces. Show that there is a linear map $\psi : W \to V$ such that $\phi \circ \psi = \mathrm{id}_W$. [4]

   (c) Let $\phi : V \to W$ be a linear map of vector spaces with transpose ϕ^T. Show that $\ker \phi = \mathrm{sol}(\mathrm{im}\, \phi^T)$. [4]

8. (a) Let V be a finite-dimensional inner product space and $U \leq V$ a subspace. What is the *orthogonal complement* $U^\perp$ of U? Show that $V = U \oplus U^\perp$. [6]

   (b) Let U be the subspace of $\mathbb{R}^4$ spanned by $(1, 1, 0, 0)$, $(0, 0, 1, 2)$ and $(0, 1, 1, 0)$. Equip $\mathbb{R}^4$ with the standard (dot) inner product.

    (i) Find an orthonormal basis for U. [6]

    (ii) Stating any results from lectures that you use, find $u \in U$ such that $\|u - (0, 1, 2, 3)\|$ is as small as possible. [6]

9. (a) Let A be the matrix

$$\begin{pmatrix} 0 & 3 & 2 & 0 \\ 3 & 0 & 0 & 2 \\ 2 & 0 & 0 & 3 \\ 0 & 2 & 3 & 0 \end{pmatrix}.$$

Find an orthogonal matrix P such that $P^{-1}AP$ is diagonal. [8]

(b) What is a *quadratic form* on a real vector space V?
What are the *rank* and *signature* of a quadratic form?
State Sylvester's Law of Inertia. [6]

(c) Let Q be the quadratic form on $\mathbb{R}^4$ given by

$$Q(x) = 3x_1x_2 + 2x_1x_3 + 2x_2x_4 + 3x_3x_4.$$

Compute the rank and signature of Q. [4]