M216: Exercise sheet 10

Warmup questions

1. Show that the following are bilinear maps:
   (a) Matrix multiplication $M_{m\times n}(\mathbb{F}) \times M_{n\times p}(\mathbb{F}) \to M_{m\times p}(\mathbb{F})$.
   (b) Evaluation $(\phi, v) \mapsto \phi(v) : L(V, W) \times V \to W$.
   (c) For $\alpha \in V^*$ and $w \in W$, define $\phi_{\alpha,w} : V \to W$ by
$$\phi_{\alpha,w}(v) = \alpha(v)w.$$
      (i) Show that each $\phi_{\alpha,w}$ is linear.
      (ii) Show that the map $t : V^* \times W \to L(V, W)$ given by $t(\alpha, w) = \phi_{\alpha,w}$ is bilinear.
2. Let $A, B \in M_{n\times n}(\mathbb{F})$ be congruent: $B = P^T AP$, for some $P \in \mathrm{GL}(n, \mathbb{F})$.
   Are the following statements true or false?
   (a) $\det A = \det B$.
   (b) A is symmetric if and only if B is symmetric.

Exam-style questions

3. Compute the rank and signature of the quadratic form $Q(x) = x_1x_2 - 4x_3x_4$ on $\mathbb{R}^4$.
4. Diagonalise the quadratic form Q on $\mathbb{R}^3$ given by $Q(x) = x_1^2 + 2x_1x_2 + 2x_2^2 + 2x_2x_3 + x_3^2$. Hence, or otherwise, compute the rank and signature of Q.
5. (2015/6 Exam) Let $\phi_A : \mathbb{R}^4 \to \mathbb{R}^4$ be the linear map corresponding to the matrix
$$A = \begin{pmatrix} 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$
   Diagonalise the quadratic form given by $Q(x) = \langle \phi_A(x), x\rangle$ and hence, or otherwise, compute its signature. [Here the inner product is dot product.]

Extra questions

6. Let V be a real finite-dimensional inner product space. The inner product is a non-degenerate symmetric bilinear form and so defines an isomorphism $\beta : V \to V^*$ by
$$\beta(v)(w) = \langle v, w\rangle,$$
   for all $v, w \in V$.
   Let us show that β relates orthogonal complements and adjoints with solution sets, annihilators and transposes:
   (a) Let $U \leq V$ and $E \leq V^*$. Prove that
$$\operatorname{ann} U = \beta(U^\perp)$$
$$\operatorname{sol} E = (\beta^{-1}(E))^\perp$$
   (b) Let $\phi \in L(V)$. Show that $\beta \circ \phi^* \circ \beta^{-1} = \phi^T$.

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M216: Exercise sheet 10—Solutions

1. (*a*) The bilinearity amounts to:

$$\begin{aligned} A(C+\lambda D) &= AC+\lambda AD \\ (A+\lambda B)C &= AC+\lambda BC, \end{aligned}$$

for all $A, B \in M_{m\times n}(\mathbb{F})$, $C, D \in M_{n\times p}(\mathbb{F})$ and $\lambda \in \mathbb{F}$. Both of these are easy to prove. For example,

$$\begin{aligned} (A(C+\lambda D))_{ij} &= \sum_{k=1}^{n} A_{ik}(C+\lambda D)_{kj} = \sum_{k=1}^{n} A_{ik}(C_{kj}+\lambda D_{kj}) \\ &= \sum_{k=1}^{n}(A_{ik}C_{kj}+\lambda A_{ik}D_{kj}) = (AC)_{ij}+\lambda(AD)_{ij} = (AC+\lambda AD)_{ij}. \end{aligned}$$

(*b*) Here, bilinearity reads

$$\begin{aligned} (\phi_1+\lambda\phi_2)(v) &= \phi_1(v)+\lambda\phi_2(v) \\ \phi(u+\lambda v) &= \phi(u)+\lambda\phi(v), \end{aligned}$$

for all $\phi, \phi_1, \phi_2 \in L(V, W)$, $u, v \in V$ and $\lambda \in \mathbb{F}$. But the first of these is simply the definition of the pointwise addition and scalar multiplication in $L(V, W)$ while the second is simply the assertion that ϕ is linear!

(*c*) (i) This comes straight from the linearity of α: for $u, v \in V$ and $\lambda \in \mathbb{F}$,

$$\phi_{\alpha,w}(u+\lambda v) = \alpha(u+\lambda v)w = \alpha(u)w+\lambda\alpha(v)w = \phi_{\alpha,w}(u)+\lambda\phi_{\alpha,w}(v).$$

(ii) Bilinearity of t amounts to:

$$\begin{aligned} \phi_{\alpha+\lambda\beta,w} &= \phi_{\alpha,w}+\lambda\phi_{\beta,w} \\ \phi_{\alpha,w_1+\lambda w_2} &= \phi_{\alpha,w_1}+\lambda\phi_{\alpha,w_2}, \end{aligned}$$

for all $\alpha, \beta \in V^*$, $w, w_1, w_2 \in W$ and $\lambda \in \mathbb{F}$. Each is proved by showing that both sides take the same values on each $v \in V$. For example:

$$\begin{aligned} \phi_{\alpha+\lambda\beta,w}(v) &= (\alpha+\lambda\beta)(v)w = \alpha(v)w+\lambda\beta(v)w \\ &= \phi_{\alpha,w}(v)+\lambda\phi_{\beta,w}(v) = (\phi_{\alpha,w}+\lambda\phi_{\beta,w})(v) \end{aligned}$$

2. (*a*) This is false: let $P = \lambda I_n$, for $\lambda \in \mathbb{F}$. Then $B = \lambda^2 A$ so that $\det B = \lambda^{2n}\det A$.

(*b*) This is true: if $A^T = A$ then

$$B^T = (P^TAP)^T = P^TA^TP = P^TAP = B.$$

Conversely, if $B^T = B$ we get $P^TA^TP = P^TAP$ and multiplying by P^{-1} on the right and $(P^T)^{-1}$ on the left gives $A^T = A$.

3. The fastest way to do this is to recall that $xy = \frac{1}{4}\big((x+y)^2-(x-y)^2\big)$ so that

$$x_1x_2-4x_3x_4 = \tfrac{1}{4}(x_1+x_2)^2-\tfrac{1}{4}(x_1-x_2)^2-(x_3+x_4)^2+(x_3-x_4)^2.$$

Moreover, the four linear functionals $x_1 \pm x_2, x_3 \pm x_4$ are linearly independent: one way to see this is that $x_1 \pm x_2 = 0 = x_3 \pm x_4$ forces each $x_i = 0$ so that Corollary 5.8 applies.
Now two squares appear positively and two negatively giving signature $(2, 2)$ and so rank 4.

4. Q has polarisation $B = B_A$ where

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

so that $B(x, y) = x_1y_1 + x_1y_2 + x_2y_1 + 2x_2y_2 + x_2y_3 + x_3y_2 + x_3y_3$.
There are several ways to get a diagonalising basis:

(i) Compute eigenvectors of A: do whatever you have to do to see that A has eigenvalues $3, 1, 0$ with eigenvectors $v_1 = (1, 2, 1)$, $v_2 = (1, 0, -1)$ and $v_3 = (1, -1, 1)$. This is a diagonalising basis with respect to which B has matrix

$$\begin{pmatrix} 18 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The i-th diagonal term is the i-th eigenvalue times the norm-squared of v_i.

(ii) Exploit the zero in the $(1, 3)$ slot: note that

$$Q(e_1) = Q(e_3) = 1, \qquad B(e_1, e_3) = 0$$

so that we just need to find y with

$$\begin{aligned} 0 &= B(e_1, y) = y_1 + y_2 \\ 0 &= B(e_3, y) = y_2 + y_3. \end{aligned}$$

Clearly $y = (1, -1, 1)$ does the job with $B(y, y) = 0$. Thus e_1, e_3, y are a diagonalising basis with matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0. \end{pmatrix}$$

Either way, we see that the signature is $(2, 0)$ and so the rank is 2.
We could see this also by writing Q directly as a sum of squares: notice that $Q(x) = (x_1 + x_2)^2 + (x_2 + x_3)^2$ and $x_1 + x_2, x_2 + x_3$ are linearly independent since they are not scalar multiples of each other..

5. First let us compute $Q(x) = 2x_1x_2 + x_1x_3 + x_2x_4 + 2x_3x_4$ with polarisation B whose matrix with respect to the standard basis is

$$\begin{pmatrix} 0 & 1 & \frac{1}{2} & 0 \\ 1 & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & 1 \\ 0 & \frac{1}{2} & 1 & 0 \end{pmatrix} = \tfrac{1}{2}(A + A^T).$$

Thus $B(x, y) = x_1y_2 + x_2y_1 + x_3y_4 + x_4y_3 + \frac{1}{2}(x_1y_3 + x_3y_1 + x_2y_4 + x_4y_2)$.
As usual, there are different ways to find a diagonalising basis:

(i) Find eigenvectors of this symmetric matrix: you should get eigenvalues $\pm\frac{1}{2}, \pm\frac{3}{2}$ with eigenvectors forming a diagonalising basis

$$\begin{aligned} v_1 &= (1, 1, -1, -1) \\ v_2 &= (1, -1, 1, -1) \\ v_3 &= (1, 1, 1, 1) \\ v_4 &= (1, -1, -1, 1) \end{aligned}$$

with respect to which B has matrix

$$\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & -6 \end{pmatrix}.$$

(ii) Use a little ingenuity: observe that all those zeros tell us that, with $U = \operatorname{span}\{e_1, e_4\}$ and $W = \operatorname{span}\{e_2, e_3\}$, Q, and so B, vanishes on U and W. So, for any $u \in U$, $w \in W$,

$$Q(u + w) = 2B(u, w), \qquad Q(u - w) = -2B(u, w), \qquad B(u + w, u - w) = 0.$$

Inspired by this, set $v_1 = (1, 1, 0, 0) = e_1 + e_2$ and $v_2 = (1, -1, 0, 0)$ to get $Q(v_1) = 2 = -Q(v_2)$ and $B(v_1, v_2) = 0$. Now seek $y \in \mathbb{R}^4$ with

$$\begin{aligned} 0 &= B(v_1, y) = y_2 + y_1 + \tfrac{1}{2}(y_3 + y_4) \\ 0 &= B(v_2, y) = y_2 - y_1 + \tfrac{1}{2}(y_3 - y_4) \end{aligned}$$

One possibility is $y = (1, 1, -2, -2)$ with $Q(y) = 6$ so put $v_3 = y = (1, 1, 2, -2)$ and seek another y that additionally has $0 = B(v_3, y) = y_2 + y_1 - 2y_4 - 2y_3 + \frac{1}{2}(y_3 - 2y_1 + y_4 - 2y_2)$

$$0 = B(v_3, y) = y_2 + y_1 - 2y_4 - 2y_3 + \tfrac{1}{2}(y_3 - 2y_1 + y_4 - 2y_2) == -\tfrac{3}{2}(y_4 + y_3).$$

The solution is $y = (-1, 1, 2, -2)$ with $Q(y) = -14$. So put $v_4 = (-1, 1, 2, -2)$ to get a diagonalising basis with matrix

$$\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & -14 \end{pmatrix}.$$

We can check our answer: let P have the v_j as columns and check that $P^T \frac{1}{2}(A + A^T)P$ is diagonal:

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & 1 & -2 & -2 \\ -1 & 1 & 2 & -2 \end{pmatrix} \begin{pmatrix} 0 & 1 & \frac{1}{2} & 0 \\ 1 & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & 1 \\ 0 & \frac{1}{2} & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & -1 \\ 1 & -1 & 1 & 1 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & -2 & -2 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & -14 \end{pmatrix}$$

Whatever diagonalising basis we find, we see that Q has signature $(2, 2)$ and so rank 4.

6. (a) Let $\alpha \in V^*$. Then $\alpha = \beta(w)$, for some $w \in V$, and $\alpha \in \operatorname{ann} U$ if and only if $\alpha(u) = 0$, for all $u \in U$, if and only if $\beta(w)(u) = 0$, for all $u \in U$, or, equivalently, $\langle w, u\rangle = 0$, for all $u \in U$, that is $w \in U^\perp$.
Let $v \in V$. For $\alpha \in V^*$, note that $\langle \beta^{-1}(\alpha), v\rangle = \beta(\beta^{-1}(\alpha))(v) = \alpha(v)$. Thus $v \in (\beta^{-1}(E))^\perp$ if and only if $\alpha(v) = \langle \beta^{-1}(\alpha), v\rangle = 0$, for all $\alpha \in E$, that is, $v \in \operatorname{sol} E$.

(b) We prove that $\beta \circ \phi^* = \phi^T \circ \beta$. So let $v \in V$. We want to show that

$$\beta(\phi^*(v)) = \phi^T(\beta(v)),$$

or, equivalently, for all $w \in V$,

$$\beta(\phi^*(v))(w) = \phi^T(\beta(v))(w).$$

However, the left hand side of this is

$$\langle \phi^*(v), w\rangle = \langle v, \phi(w)\rangle = \beta(v)(\phi(w)) = (\beta(v) \circ \phi)(w) = (\phi^T(\beta(v)))(w),$$

as required.

Note that, in all of this, the only place where we used that V was finite-dimensional was to know that β was an isomorphism. Thus our arguments apply in infinite-dimensional settings (such as Hilbert spaces) where β is also an isomorphism.