M216: Exercise sheet 10

Warmup questions

1. Show that the following are bilinear maps:
   (*a*) Matrix multiplication $M_{m\times n}(\mathbb{F}) \times M_{n\times p}(\mathbb{F}) \to M_{m\times p}(\mathbb{F})$.
   (*b*) Evaluation $(\phi, v) \mapsto \phi(v) : L(V,W) \times V \to W$.
   (*c*) For $\alpha \in V^*$ and $w \in W$, define $\phi_{\alpha,w} : V \to W$ by
   $$\phi_{\alpha,w}(v) = \alpha(v)w.$$
   (i) Show that each $\phi_{\alpha,w}$ is linear.
   (ii) Show that the map $t : V^* \times W \to L(V,W)$ given by $t(\alpha,w) = \phi_{\alpha,w}$ is bilinear.

2. Let $B : V \times V \to \mathbb{F}$ be a symmetric bilinear form with diagonalising basis $v_1, \dots, v_n$. Suppose that, for some v_i, $1 \le i \le n$, we have $B(v_i, v_i) = 0$. Prove that $v_i \in \operatorname{rad} B$.

3. Let $B : V \times V \to \mathbb{F}$ be a real symmetric bilinear form with diagonalising basis $v_1, \dots, v_n$. Show that B is positive definite if and only if $B(v_i, v_i) > 0$, for all $1 \le i \le n$.

4. Let $A, B \in M_{n\times n}(\mathbb{F})$ be congruent: $B = P^T AP$, for some $P \in \mathrm{GL}(n, \mathbb{F})$.
   Are the following statements true or false?
   (*a*) $\det A = \det B$.
   (*b*) A is symmetric if and only if B is symmetric.

Rank and signature

5. Let $B = B_A : \mathbb{R}^4 \times \mathbb{R}^4 \to \mathbb{R}$ where
   $$A = \begin{pmatrix} 0 & 2 & 1 & 0 \\ 2 & 0 & 0 & 1 \\ 1 & 0 & 0 & 2 \\ 0 & 1 & 2 & 0 \end{pmatrix}.$$
   Diagonalise B and hence, or otherwise, compute its signature.

6. Diagonalise the symmetric bilinear form $B : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ given by $B(x,y) = x_1y_1 + x_1y_2 + x_2y_1 + 2x_2y_2 + x_2y_3 + x_3y_2 + x_3y_3$.
   Hence, or otherwise, compute the rank and signature of B.

7. Compute the rank and signature of the quadratic form $Q(x) = x_1x_2 - 4x_3x_4$ on $\mathbb{R}^4$.

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M216: Exercise sheet 10—Solutions

1. (*a*) The bilinearity amounts to:

$$A(C + \lambda D) = AC + \lambda AD$$
$$(A + \lambda B)C = AC + \lambda BC,$$

for all $A, B \in M_{m\times n}(\mathbb{F})$, $C, D \in M_{n\times p}(\mathbb{F})$ and $\lambda \in \mathbb{F}$. Both of these are easy to prove. For example,

$$(A(C + \lambda D))_{ij} = \sum_{k=1}^{n} A_{ik}(C + \lambda D)_{kj} = \sum_{k=1}^{n} A_{ik}(C_{kj} + \lambda D_{kj})$$
$$= \sum_{k=1}^{n}(A_{ik}C_{kj} + \lambda A_{ik}D_{kj}) = (AC)_{ij} + \lambda(AD)_{ij} = (AC + \lambda AD)_{ij}.$$

(*b*) Here, bilinearity reads

$$(\phi_1 + \lambda\phi_2)(v) = \phi_1(v) + \lambda\phi_2(v)$$
$$\phi(u + \lambda v) = \phi(u) + \lambda\phi(v),$$

for all $\phi, \phi_1, \phi_2 \in L(V, W)$, $u, v \in V$ and $\lambda \in \mathbb{F}$. But the first of these is simply the definition of the pointwise addition and scalar multiplication in $L(V, W)$ while the second is simply the assertion that ϕ is linear!

(*c*) (i) This comes straight from the linearity of α: for $u, v \in V$ and $\lambda \in \mathbb{F}$,

$$\phi_{\alpha,w}(u + \lambda v) = \alpha(u + \lambda v)w = \alpha(u)w + \lambda\alpha(v)w = \phi_{\alpha,w}(u) + \lambda\phi_{\alpha,w}(v).$$

(ii) Bilinearity of t amounts to:

$$\phi_{\alpha+\lambda\beta,w} = \phi_{\alpha,w} + \lambda\phi_{\beta,w}$$
$$\phi_{\alpha,w_1+\lambda w_2} = \phi_{\alpha,w_1} + \lambda\phi_{\alpha,w_2},$$

for all $\alpha, \beta \in V^*$, $w, w_1, w_2 \in W$ and $\lambda \in \mathbb{F}$. Each is proved by showing that both sides take the same values on each $v \in V$. For example:

$$\phi_{\alpha+\lambda\beta,w}(v) = (\alpha + \lambda\beta)(v)w = \alpha(v)w + \lambda\beta(v)w$$
$$= \phi_{\alpha,w}(v) + \lambda\phi_{\beta,w}(v) = (\phi_{\alpha,w} + \lambda\phi_{\beta,w})(v)$$

2. In this case, we have $B(v_i, v_j) = 0$, for all $1 \leq j \leq n$. So, if $v \in V$, write $v = \sum_j \lambda_j v_j$ and then

$$B(v_i, v) = \sum_j \lambda_j B(v_i, v_j) = 0.$$

Otherwise said, $v_i \in \operatorname{rad} B$.

3. If B is positive definite, then $B(v, v) > 0$ for any non-zero $v \in V$ and so, in particular, each $B(v_i, v_i) > 0$.
Conversely, suppose that each $B(v_i, v_i) > 0$ and let $v \in V$. Write $v = \lambda_1 v_1 + \cdots + \lambda_n v_n$ and compute:

$$B(v, v) = B(\sum_i \lambda_i v_i, \sum_j \lambda_j v_j) = \sum_{i,j} \lambda_i\lambda_j B(v_i, v_j) = \sum_i \lambda_i^2 B(v_i, v_i).$$

This last is non-negative and vanishes if and only if each $\lambda_i^2 B(v_i, v_i) = 0$, or, equivalently, $\lambda_i = 0$. Thus B is positive definite.

4. (*a*) This is false: let $P = \lambda I_n$, for $\lambda \in \mathbb{F}$. Then $B = \lambda^2 A$ so that $\det B = \lambda^{2n} \det A$.

   (*b*) This is true: if $A^T = A$ then

$$B^T = (P^T A P)^T = P^T A^T P = P^T A P = B.$$

   Conversely, if $B^T = B$ we get $P^T A^T P = P^T A P$ and multiplying by P^{-1} on the right and $(P^T)^{-1}$ on the left gives $A^T = A$.

5. We need to start with v_1 with $B(v_1, v_1) \neq 0$. Those diagonal zeros say that none of the standard basis will do so let us try $v_1 = (1, 1, 0, 0)$ for which $B(v_1, v_1) = 4$. Now seek v_2 among the y with

$$0 = B(v_1, y) = \begin{pmatrix} 1 & 1 & 0 & 0 \end{pmatrix} A\mathbf{y} = \begin{pmatrix} 2 & 2 & 1 & 1 \end{pmatrix} \mathbf{y} = 2y_1 + 2y_2 + y_3 + y_4.$$

We take $v_2 = (0, 0, 1, -1)$ with

$$B(v_2, y) = \begin{pmatrix} 0 & 0 & 1 & -1 \end{pmatrix} A\mathbf{y} = \begin{pmatrix} 1 & -1 & -2 & 2 \end{pmatrix} \mathbf{y} = y_1 - y_2 - 2y_3 + 2y_4.$$

Then $B(v_2, v_2) = -4$ and we seek v_3 among the y with $B(v_1, y) = B(v_2, y) = 0$, that is:

$$\begin{aligned} 2y_1 + 2y_2 + y_3 + y_4 &= 0 \\ y_1 - y_2 - 2y_3 + 2y_4 &= 0. \end{aligned}$$

One solution is $v_3 = (-3, 5, -4, 0)$ with

$$B(v_3, y) = \begin{pmatrix} -3 & 5 & -4 & 0 \end{pmatrix} A\mathbf{y} = 3\begin{pmatrix} 2 & -2 & -1 & -1 \end{pmatrix} \mathbf{y} = 3(2y_1 - 2y_2 - y_3 - y_4).$$

Thus $B(v_3, v_3) = -36$ and we need to find $v_4 = y$ with $B(v_1, y) = B(v_2, y) = B(v_3, y) = 0$:

$$\begin{aligned} 2y_1 + 2y_2 + y_3 + y_4 &= 0 \\ y_1 - y_2 - 2y_3 + 2y_4 &= 0 \\ 2y_1 - 2y_2 - y_3 - y_4 &= 0. \end{aligned}$$

A solution is $v_4 = (0, 4, -5, -3)$ with $B(v_4, v_4) = 36$.
We now have a diagonalising basis with $B(v_i, v_i) = 4, -4, -36, 36$ so B has signature $(2, 2)$ and so has rank 4.
After all this linear equation solving it is probably good to check our answer: let P have the v_j as columns and check that $P^T A P$ is diagonal:

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ -3 & 5 & -4 & 0 \\ 0 & 4 & -5 & -3 \end{pmatrix} \begin{pmatrix} 0 & 2 & 1 & 0 \\ 2 & 0 & 0 & 1 \\ 1 & 0 & 0 & 2 \\ 0 & 1 & 2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & -3 & 0 \\ 1 & 0 & 5 & 4 \\ 0 & 1 & -4 & -5 \\ 0 & -1 & 0 & -3 \end{pmatrix} = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & -36 & 0 \\ 0 & 0 & 0 & 36 \end{pmatrix}$$

6. $B = B_A$ where

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

Let us exploit the zero in the $(1,3)$ slot: note that

$$B(e_1, e_1) = B(e_3, e_3) = 1, \qquad B(e_1, e_3) = 0$$

so that we just need to find y with

$$0 = B(e_1, y) = y_1 + y_2$$
$$0 = B(e_3, y) = y_2 + y_3.$$

Clearly $y = (1, -1, 1)$ does the job with $B(y, y) = 0$. Thus e_1, e_3, y are a diagonalising basis with matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0. \end{pmatrix}$$

Either way, we see that the signature is $(2,0)$ and so the rank is 2.

7. The fastest way to do this is to recall that $xy = \frac{1}{4}\big((x+y)^2 - (x-y)^2\big)$ so that

$$x_1x_2 - 4x_3x_4 = \tfrac{1}{4}(x_1 + x_2)^2 - \tfrac{1}{4}(x_1 - x_2)^2 - (x_3 + x_4)^2 + (x_3 - x_4)^2.$$

Moreover, the four linear functionals $x_1 \pm x_2, x_3 \pm x_4$ are linearly independent: one way to see this is that $x_1 \pm x_2 = 0 = x_3 \pm x_4$ forces each $x_i = 0$ so that Corollary 5.7 applies.
Now two squares appear positively and two negatively giving signature $(2,2)$ and so rank 4.