

M216: Exercise sheet 9

Warmup questions

1. Let $\alpha_1, \dots, \alpha_k$ span $E \leq V^*$. Show that

$$\text{sol } E = \bigcap_{i=1}^k \ker \alpha_i.$$

2. Let $U \leq V$. Show that $\text{ann } U \leq V^*$.
3. Let V be finite-dimensional and $U \leq V$. Show that

$$\dim \text{ann } U + \dim U = \dim V.$$

Homework

4. Prove at least one of the following assertions:

(a) Let $E, F \leq V^*$. Then

$$\begin{aligned} \text{sol}(E + F) &= (\text{sol } E) \cap (\text{sol } F) \\ (\text{sol } E) + (\text{sol } F) &\leq \text{sol}(E \cap F) \end{aligned}$$

with equality if V is finite-dimensional.

(b) Let $U, W \leq V$. Then

$$\begin{aligned} \text{ann}(U + W) &= (\text{ann } U) \cap (\text{ann } W) \\ (\text{ann } U) + (\text{ann } W) &\leq \text{ann}(U \cap W) \end{aligned}$$

with equality if V is finite-dimensional.

5. Let $\phi \in L(V, W)$ be a linear map of vector spaces. Show that

$$\begin{aligned} \ker \phi^T &= \text{ann}(\text{im } \phi) \\ \text{im } \phi^T &\leq \text{ann}(\ker \phi) \end{aligned}$$

with equality if V, W are finite-dimensional.

Extra questions

6. Let $U \leq V$ and let $\iota : U \rightarrow V$ be the inclusion map (so that $\iota(u) = u$, for all $u \in U$) and $q : V \rightarrow V/U$ the quotient map.

(a) Show that $\iota^T : V^* \rightarrow U^*$ is the restriction map: thus $\iota^T(\alpha) = \alpha|_U$ with kernel $\text{ann } U$.

If V is finite-dimensional, show that ι^T is surjective and deduce that $V^*/\text{ann } U \cong U^*$.

(b) Show that $q^T : (V/U)^* \rightarrow V^*$ is injective with $\text{im } q^T \leq \text{ann } U$. If V is finite-dimensional, show that q^T is an isomorphism $(V/U)^* \rightarrow \text{ann } U$.

7. Recall the linear injection $\text{ev} : V \rightarrow V^{**}$. For $U \leq V$, show that $\text{ev}(U) \leq \text{ann}(\text{ann } U)$ with equality if V is finite-dimensional.

Please hand in at 4W level 1 by NOON on Friday 7th December

Home page: <http://go.bath.ac.uk/ma20216>

M216: Exercise sheet 9—Solutions

1. Let $v \in \text{sol } E$ so that $\alpha(v) = 0$, for all $\alpha \in E$. Then, in particular, each $\alpha_i(v) = 0$ so that $v \in \ker \alpha_i$, for $1 \leq i \leq k$. That is, $v \in \bigcap_{i=1}^k \ker \alpha_i$ and $\text{sol } E \leq \bigcap_{i=1}^k \ker \alpha_i$.
Conversely, let $v \in \bigcap_{i=1}^k \ker \alpha_i$ so that $\alpha_i(v) = 0$, for $1 \leq i \leq k$. Let $\alpha \in E$. Then $\alpha = \sum_{i=1}^k \lambda_i \alpha_i$, for some $\lambda_1, \dots, \lambda_k \in \mathbb{F}$, since the α_i span E , and

$$\alpha(v) = \sum_{i=1}^k \lambda_i \alpha_i(v) = 0$$

so that $v \in \text{sol } E$. Thus $\bigcap_{i=1}^k \ker \alpha_i \leq \text{sol } E$ and we are done.

2. Firstly, $0 \in \text{ann } U$ so $\text{ann } U \neq \emptyset$. So we just check that $\text{ann } U$ is closed under addition and scalar multiplication. Let $\alpha_1, \alpha_2 \in \text{ann } U$ and $u \in U$. Then, $\alpha_1(u) = \alpha_2(u) = 0$ so that $(\alpha_1 + \alpha_2)(u) = 0 + 0 = 0$ whence $\alpha_1 + \alpha_2 \in \text{ann } U$ also. Similarly, for $\alpha \in \text{ann } U$ and $\lambda \in \mathbb{F}$, $(\lambda\alpha)(u) = \lambda\alpha(u) = \lambda \cdot 0 = 0$ so that $\lambda\alpha \in \text{ann } U$.
Alternatively, note that restriction to U , $\alpha \mapsto \alpha|_U$ is a linear map $V^* \rightarrow U^*$ with kernel $\text{ann } U$.
3. Let v_1, \dots, v_k be a basis of U and extend to a basis v_1, \dots, v_n of V . Let v_1^*, \dots, v_n^* be the dual basis. Now observe that $\alpha \in V^*$ is in $\text{ann } U$ if and only if $\alpha(v_j) = 0$, for $1 \leq j \leq k$. Thus, writing $\alpha = \sum_{i=1}^n \alpha(v_i) v_i^*$, we see that $\alpha \in \text{ann } U$ if and only if $\alpha \in \text{span}\{v_i^* \mid k+1 \leq i \leq n\}$. Thus $\text{ann } U = \text{span}\{v_i^* \mid k+1 \leq i \leq n\}$ so that

$$\dim \text{ann } U = n - k = \dim V - \dim U.$$

4. (a) $E, F \leq E + F$ so $\text{sol}(E + F) \leq \text{sol } E, \text{sol } F$ whence $\text{sol}(E + F) \leq (\text{sol } E) \cap (\text{sol } F)$.
Conversely, if $v \in (\text{sol } E) \cap (\text{sol } F)$ then $\alpha(v) = \beta(v) = 0$, for all $\alpha \in E$ and $\beta \in F$. Thus, for $\alpha + \beta \in E + F$, $(\alpha + \beta)(v) = 0 + 0 = 0$ so that $v \in \text{sol}(E + F)$. We conclude that $(\text{sol } E) \cap (\text{sol } F) \leq \text{sol}(E + F)$ and so $(\text{sol } E) \cap (\text{sol } F) = \text{sol}(E + F)$.
For the second statement, $E \cap F \leq E, F$ so that $\text{sol } E, \text{sol } F \leq \text{sol}(E \cap F)$ whence $(\text{sol } E) + (\text{sol } F) \leq \text{sol}(E \cap F)$ by Proposition 2.1(2). For equality when V is finite-dimensional, we show that both subspaces have the same dimension using the first part, the formula for $\text{sol } E$ and the dimension formula¹. The dimension formula gives

$$\begin{aligned} \dim((\text{sol } E) + (\text{sol } F)) &= \dim \text{sol } E + \dim \text{sol } F - \dim((\text{sol } E) \cap (\text{sol } F)) \\ &= \dim \text{sol } E + \dim \text{sol } F - \dim \text{sol}(E + F), \end{aligned}$$

using the first part,

$$\begin{aligned} &= \dim V - \dim E + \dim V - \dim F - (\dim V - \dim(E + F)) \\ &= \dim V - \dim(E \cap F), \end{aligned}$$

by the dimension formula again,

$$= \dim \text{sol}(E \cap F).$$

- (b) First we note that if $X \leq Y \leq V$ then $\text{ann } Y \leq \text{ann } X$: if $\alpha \in \text{ann } Y$, then $\alpha|_Y = 0$ and so, in particular, $\alpha|_X = 0$, that is $\alpha \in \text{ann } X$.

We now put this to work: $U, W \leq U + W$ so $\text{ann}(U + W) \leq \text{ann } U, \text{ann } W$ whence $\text{ann}(U + W) \leq (\text{ann } U) \cap (\text{ann } W)$. For the converse, if $\alpha \in (\text{ann } U) \cap (\text{ann } W)$ we have $\alpha|_U = 0$ and $\alpha|_W = 0$. So if $v = u + w \in U + W$ then $\alpha(v) = \alpha(u) + \alpha(w) = 0 + 0 = 0$ so that $v \in \text{ann}(U + W)$. Thus $\text{ann}(U + W) = (\text{ann } U) \cap (\text{ann } W)$.

¹If $X, Y \leq W$ then $\dim(X + Y) + \dim(X \cap Y) = \dim X + \dim Y$.

For the second statement, $U \cap W \leq U, W$ so that $\text{ann } U, \text{ann } W \leq \text{ann}(U \cap W)$ and then $(\text{ann } U) + (\text{ann } W) \leq \text{ann}(U \cap W)$ by Proposition 2.1(2). For equality when V is finite-dimensional, we argue as in part (a). The dimension formula says

$$\begin{aligned} \dim((\text{ann } U) + (\text{ann } W)) &= \dim \text{ann } U + \dim \text{ann } W - \dim((\text{ann } U) \cap (\text{ann } W)) \\ &= \dim \text{ann } U + \dim \text{ann } W - \dim \text{ann}(U + W), \end{aligned}$$

using the first part,

$$\begin{aligned} &= \dim V - \dim U + \dim V - \dim W - (\dim V - \dim(U + W)) \\ &= \dim V - \dim(U \cap W), \end{aligned}$$

by the dimension formula again,

$$= \dim \text{ann}(U \cap W).$$

Notice that the arguments for part (b) are essentially identical to those for part (a): the key points are that ann and sol reverse inclusions and take subspaces to ones of complementary dimension.

5. Let $\alpha \in W^*$. Then $\alpha \in \ker \phi^T$ if and only if $\alpha \circ \phi = 0$ if and only if $\alpha(\text{im } \phi) = \{0\}$, that is $\alpha \in \text{ann}(\text{im } \phi)$. Thus $\ker \phi^T = \text{ann}(\text{im } \phi)$.

For the second statement, suppose that $\beta \in \text{im } \phi^T$ so that $\beta = \phi^T(\alpha) = \alpha \circ \phi$, for some $\alpha \in W^*$. Then if $v \in \ker \phi$, $\beta(v) = \alpha(\phi(v)) = 0$ so that $\beta \in \text{ann}(\ker \phi)$. Thus $\text{im } \phi^T \leq \text{ann}(\ker \phi)$.

For equality when V is finite-dimensional, recall that we already know from lectures that $\text{rank } \phi = \text{rank } \phi^T$ from which we see from rank-nullity that

$$\dim \text{im } \phi^T = \text{rank } \phi = \dim V - \dim \ker \phi = \dim \text{ann}(\ker \phi).$$

6. (a) For $\alpha \in V^*$ and $u \in U$, $\iota^T(\alpha)(u) = \alpha(\iota(u)) = \alpha(u) = \alpha|_U(u)$. Thus $\iota^T(\alpha) = \alpha|_U$ and ι^T is the restriction map. Now $\ker \iota^T = \{\alpha \in V^* \mid \alpha|_U = 0\} = \text{ann } U$.

Proposition 2.11 tells us² that any $\beta \in U^*$ is the restriction of some $\alpha \in V^*$ so that ι^T surjects: $\text{im } \iota^T = U^*$. Thus, the First Isomorphism Theorem, applied to ι^T , tells us that

$$V^* / \text{ann } U = V^* / \ker \iota^T \cong \text{im } \iota^T = U^*.$$

This gives us another approach to Question 3.

- (b) All we need to know about q is that it is a linear surjection with kernel U . Then, by Question 5, $\ker q^T = \text{ann}(\text{im } q) = \text{ann } V/U = \{0\}$ (any $\alpha \in (V/U)^*$ that vanishes on V/U is zero by definition!) so that q^T injects. Moreover, Question 5 tells us that $\text{im } q^T \leq \text{ann}(\ker q) = \text{ann } U$ with equality when V is finite-dimensional. Thus, in that case, q^T is a linear bijection $(V/U)^* \rightarrow \text{ann } U$ and so an isomorphism.

7. This is just a matter of not panicking! Let $f \in \text{ev}(U)$ so that $f = \text{ev}(u)$, for some $u \in U$. Let $\alpha \in \text{ann } U$. We need $f(\alpha) = 0$. But

$$f(\alpha) = \text{ev}(u)(\alpha) = \alpha(u) = 0,$$

since $\alpha \in \text{ann } U$.

When V is finite-dimensional, we know that ev is an isomorphism so that $\dim \text{ev}(U) = \dim U$. Meanwhile

$$\dim(\text{ann}(\text{ann } U)) = \dim V^* - \dim \text{ann } U = \dim V - (\dim V - \dim U) = \dim U$$

so that $\text{ev}(U)$ and $\text{ann}(\text{ann } U)$ have the same dimension and so coincide.

²This is where we use that V is finite-dimensional.