M216: Exercise sheet 8

Warmup questions

1. Find a unitary matrix P such that $P^{-1}AP$ is diagonal where

$$A = \begin{pmatrix} 1 & 2-2i \\ 2+2i & 3 \end{pmatrix}.$$

2. Let $\phi \in L(V)$ be self-adjoint with eigenvalues $\lambda_1, \dots, \lambda_n$. What are the singular values of ϕ?

3. Let $\phi = \phi_A : \mathbb{R}^2 \to \mathbb{R}^2$ where

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

What are the singular values of ϕ?

Homework

4. Find an orthogonal matrix P such that $P^{-1}AP$ is diagonal where

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 2 \end{pmatrix}.$$

5. Let $U \leq V$ and $v \in V \setminus U$. Show that there is $\alpha \in V^*$ such that α is zero on U but $\alpha(v) \neq 0$.
Hint: Apply theorem 5.4 to V/U.

Extra questions

6. (2016/7 Exam) Let V be the space of all real-valued polynomials $p(t)$ of degree ≤ 2016. For any real numbers a_j, $0 \leq j \leq 2016$, show that there is a $p \in V$ such that, for all $q \in V$,

$$\int_0^1 p(t)q(t)\,dt = a_0q(0) + a_1q(1) + \cdots + a_{2016}q(2016).$$

[You may assume that the left side defines an inner product on V.]

7. Let V be a vector space over $\mathbb{F}$. For $v \in V$, define $\mathrm{ev}(v) : V^* \to \mathbb{F}$ by

$$\mathrm{ev}(v)(\alpha) = \alpha(v).$$

(a) Show that $\mathrm{ev}(v)$ is linear so that $\mathrm{ev}(v) \in V^{**}$.
(b) We therefore have a map $\mathrm{ev} : V \to V^{**}$. Show that ev is linear.
(c) Show that ev is injective.
(d) Deduce that if V is finite-dimensional then $\mathrm{ev} : V \to V^{**}$ is an isomorphism.

Please hand in at 4W level 1 by NOON on Friday November 30th

Home page: `http://go.bath.ac.uk/ma20216`

M216: Exercise sheet 8—Solutions

1. A is Hermitian so we know that ϕ_A has an orthonormal basis of eigenvectors that we can use for the columns of P. The characteristic polynomial of A is $\lambda^2 - 4\lambda - 5$ with roots 5 and -1. Corresponding eigenvectors are $(1, 1+i)$ and $(1-i, -1)$ both of which have norm $\sqrt{3}$ so that

$$P = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1-i \\ 1+i & -1 \end{pmatrix}.$$

You may get a different answer since if v is a unit eigenvector, so is any $e^{i\theta}v$. Thus there are lots of different possible P. However, we can check our answers by verifying $P^\dagger P = I$ and $P^\dagger AP$ is diagonal. My P is Hermitian: $P = P^\dagger$ so that

$$P^\dagger P = P^2 = \frac{1}{3} \begin{pmatrix} 1 & 1-i \\ 1+i & -1 \end{pmatrix} \begin{pmatrix} 1 & 1-i \\ 1+i & -1 \end{pmatrix} = I$$

while

$$\begin{aligned} P^\dagger AP &= \begin{pmatrix} 1 & 1-i \\ 1+i & -1 \end{pmatrix} \begin{pmatrix} 1 & 2-2i \\ 2+2i & 3 \end{pmatrix} \begin{pmatrix} 1 & 1-i \\ i+1 & -1 \end{pmatrix} \\ &= \frac{1}{3} \begin{pmatrix} 5 & 5(1-i) \\ -(1+i) & 1 \end{pmatrix} \begin{pmatrix} 1 & 1-i \\ i+1 & -1 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 15 & 0 \\ 0 & -3 \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ 0 & -1 \end{pmatrix}. \end{aligned}$$

2. Let ϕ have eigenvectors $u_1, \dots, u_n$ with eigenvalues $\lambda_1, \dots, \lambda_n$. Then $\phi^* \circ \phi = \phi \circ \phi$ has the same eigenvectors but eigenvalues $\lambda_1^2, \dots, \lambda_n^2$:

$$\phi \circ \phi(u_i) = \phi(\lambda_i u_i) = \lambda_i^2 u_i.$$

The singular values of ϕ are the positive square roots of the eigenvalues of $\phi^* \circ \phi$, that is $|\lambda_1|, \dots, |\lambda_n|$.

3. We want the positive square roots of the eigenvalues of $\phi_A^* \circ \phi_A = \phi_{A^T A}$. Now

$$A^T A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

so that the singular values of ϕ_A are $0, 1$.

4. First note that A is symmetric and so ϕ_A has an orthonormal basis of eigenvectors which we can use to get the columns of P.
So first find the eigenvalues of A. The characteristic polynomial is

$$\begin{aligned} (2-\lambda)(3-\lambda)(2-\lambda) - (2-\lambda) - (2-\lambda) &= (2-\lambda)((3-\lambda)(2-\lambda) - 2) \\ &= (2-\lambda)(\lambda^2 - 5\lambda + 4) = (2-\lambda)(\lambda-1)(\lambda-4) \end{aligned}$$

so that the eigenvalues are $1, 2, 4$.
Solve linear equations to get corresponding eigenvectors $(1, -1, 1)$, $(1, 0, -1)$ and $(1, 2, 1)$ with norms $\sqrt{3}, \sqrt{2}, \sqrt{6}$ respectively so P is given by

$$P = \frac{1}{\sqrt{6}} \begin{pmatrix} \sqrt{2} & \sqrt{3} & 1 \\ -\sqrt{2} & 0 & 2 \\ \sqrt{2} & -\sqrt{3} & 1 \end{pmatrix}.$$

Your columns and so your P may differ from mine by a sign but either way we can check our answer by verifying that $P^T P = I$ and then that $P^T AP$ is diagonal. In fact,

$$P^T AP = \begin{pmatrix} 1 & & \\ & 2 & \\ & & 4. \end{pmatrix}$$

5. Let $q : V \to V/U$ be the quotient map so that q is a linear surjection with kernel U (this is all we need to know about the quotient construction). Since $v \notin U$, $q(v) \neq 0$ so that, by the Sufficiency Principle (Theorem 5.4), there is $\beta \in (V/U)^*$ such that $\beta(q(v)) \neq 0$.
Let $\alpha = \beta \circ q : V \to \mathbb{F}$. This is linear, being a composition of linear maps, so $\alpha \in V^*$. Moreover, $\alpha(v) = \beta(q(v)) \neq 0$ while, if $u \in U$, $q(u) = 0$ so that $\alpha(u) = \beta(0) = 0$.

6. V is a real finite-dimensional ($\dim V = 2017$) inner product space with inner product

$$\langle p, q \rangle := \int_0^1 p(t)q(t)dt.$$

Define $\alpha : V \to \mathbb{R}$ by

$$\alpha(q) = a_0 q(0) + a_1 q(1) + \cdots + a_{2016} q(2016).$$

Then α is linear so that $\alpha \in V^*$. Now the Riesz Representation Theorem says that there is $p \in V$ such that $\alpha(q) = \langle p, q \rangle$, for all $q \in V$. This is the p we were asked to find.

7. This is a case of thinking carefully what each statement means after which it will be very easy to prove.

   (a) To see that $\mathrm{ev}(v) : V^* \to \mathbb{F}$ is linear, we must show that

   $$\mathrm{ev}(v)(\alpha + \lambda\beta) = \mathrm{ev}(v)(\alpha) + \lambda\,\mathrm{ev}(v)(\beta),$$

   for all $\alpha, \beta \in V^*$ and $\lambda \in \mathbb{F}$. Using the definition of $\mathrm{ev}(v)$, this reads

   $$(\alpha + \lambda\beta)(v) = \alpha(v) + \lambda\beta(v)$$

   which is exactly the definition of the (pointwise) addition and scalar multiplication in V^*.

   (b) Linearity of $\mathrm{ev} : V \to V^{**}$ means that for $v, w \in V$ and $\lambda \in \mathbb{F}$, we have

   $$\mathrm{ev}(v + \lambda w) = \mathrm{ev}(v) + \lambda\,\mathrm{ev}(w).$$

   This is supposed to be equality of elements of V^{**}, that is to say, equality of two functions on V^*. This holds when the two functions give the same answers on any $\alpha \in V^*$ so we need

   $$\mathrm{ev}(v + \lambda w)(\alpha) = \mathrm{ev}(v)(\alpha) + \lambda\,\mathrm{ev}(w)(\alpha).$$

   However, using the definition of ev, this reads

   $$\alpha(v + \lambda w) = \alpha(v) + \lambda\alpha(w)$$

   which is true since α is linear!

   (c) ev is injective if and only if $\ker \mathrm{ev} = \{0\}$. Let $v \in \ker \mathrm{ev}$. Thus $\mathrm{ev}(v) = 0 \in V^{**}$, the zero functional on V^*. Otherwise said, $\mathrm{ev}(v)(\alpha) = 0$, for all $\alpha \in V^*$, or equivalently, $\alpha(v) = 0$, for all $\alpha \in V^*$. But the Sufficiency Principle now forces $v = 0$ so that ev injects.

   (d) If v is finite-dimensional, $\dim V = \dim V^* = \dim V^{**}$ so that ev is an isomorphism by rank-nullity since we have just seen that it injects.